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Jens Høyrup

Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought I

Til Sara og Janne

Abstract

Through a broad structural analysis and a close reading of Old Babylonian mathematical "procedure texts" dealing mainly with problems of the second degree it is shown that Old Babylonian "algebra" was neither a "rhetorical algebra" dealing with numbers and arithmetical relations between numbers nor built on a set of fixed algorithmic procedures. Instead, the texts must be read as "naive" prescriptions for geometric analysis–naive in the sense that the results are seen by immediate intuition to be correct, but the question of correctness never raised–dealing with measured or measurable but unknown line segments, and making use of a set of operations and techniques different in structure from that of arithmetical algebra.

The investigation involves a thorough discussion and re-interpretation of the technical terminology of Old Babylonian mathematics, elucidates many terms and procedures which have up to now been enigmatic, and makes many features stand out which had not been noticed before.

The second-last chapter discusses the metamathematical problem, whether and to which extent we are then entitled to speak of an Old Babylonian *algebra*; it also takes up the over-all implications of the investigation for the understanding of Old Babylonian patterns of thought. It is argued that these are not *mythopoeic* in the sense of H. and H. A. Frankfort, nor *savage* or *cold* in a Lévi-Straussian sense, nor however as abstract and modern as current interpretations of the mathematical texts would have them to be.

The last chapter investigates briefly the further development of Babylonian "algebra" through the Seleucid era, demonstrating a clear arithmetization of the patterns of mathematical thought, the possible role of Babylonian geometrical analysis as inspiration for early Greek geometry, and the legacy of Babylonian "algebraic" thought to Medieval Islamic algebra.

Introduction

The following contains an account of a broad investigation of the terminology, methods, and patterns of thought of Old Babylonian so-called *algebra*. I have been engaged in this investigation for some years, and circulated a preliminary and fairly unreadable account in 1984, of which the item (Høyrup 1985) in the bibliography of the present article is a slightly corrected reprint. I have also presented the progress of the project in the four Workshops on Concept Development in Babylonian Mathematics held at the Seminar für Vorderasiatische Altertumskunde der Freien Universität Berlin in 1983, 1984, 1985 and 1988, and included summaries of some of my results–without the detailed arguments–in various contexts where they were relevant.

This article is then meant to cover my results coherently and to give the details of the argument, without renouncing completely on readability. Admittedly, the article contains many discussions of philological details which will hardly be understandable to historians of mathematics without special assyriological training, but which were necessary if philological specialists should be able to evaluate my results; I hope the non-specialist will not be too disturbed by these stumbling–stones. On the other hand many points which are trivial to the assyriologist are included in order to make it clear to the non–specialist why current interpretations and translations are only reliable up to a certain point, and why the complex discussions of terminological structure and philological details are at all necessary. I apologize to whoever will find them boring and superfluous.

It is a most pleasant duty to express my gratitude to all those who have assisted me over the years,–especially Dr. Bendt Alster, Dr. Aage Westenholz and Dr. Mogens Trolle Larsen of Copenhagen University, and to Professor, Dr. Hans Nissen, Professor, Dr. Johannes Renger, Dr. Robert Englund, and Dr. Kilian Butz of Freie Universität Berlin, together with all participants in the Berlin Workshops, not least the indefatigable Professor Jöran Friberg of Göteborg University, Professor Marvin Powell of Northern Illinois University and Professor, Dr. Wolfgang Lefèvre. Special thanks are due to Professor, Dr. von Soden for giving always in the briefest possible delay kind but yet precise criticism of every preliminary and unreadable paper I sent him, and for adding always his gentle advice and encouragement.

Everybody who followed the Berlin Workshop will know that Dr. Peter Damerow of the Max-Planck-Institut für Bildungsforschung, Berlin, deserves the greatest gratitude of all, to which I can add my personal experience as made over the last six years.

The intelligent reader will easily guess who remains responsible for all errors.

I dedicate the work to my daughters Sara and Janne, for reasons which have nothing to do with mathematics, Babylonia or Assyriology, but much with our common history over the years.

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Abbreviations

Detailed bibliographic information will be found in the bibliography.

ABZ	Assyrisch-babylonische Zeichenliste (=Borger 1978)
AHw	Akkadisches Handwörterbuch (=von Soden 1965)
BiOr	Bibliotheca Orientalis
CAD	Chicago Assyrian Dictionary
GAG	Grundriss der akkadischen Grammatik (=von Soden 1952)
GEL	A Greek-English Lexicon (=Liddell – Scott 1968)

HAHw	Hebräisches und Aramäisches Handwörterbuch (=Gesenius 1915)
JCS	Journal of Cuneiform Studies
JNES	Journal of Near Eastern Studies
MCT	Mathematical Cuneiform Texts (=Neugebauer – Sachs 1945)
MEA	Manuel d'épigraphie akkadienne (=Labat 1963)
MKT	Mathematische Keilschrift-Texte, I–III (=Neugebauer 1935)
RA	Revue d'Assyriologie et d'Archéologie Orientale
ŠL	Šumerisches Lexikon, I–III (=Deimel 1925)
SLa	The Sumerian Language (=Thomsen 1984)
TMB	Textes mathématiques Babyloniens (=Thureau-Dangin 1938)
TMS	Textes mathématiques de Suse (=Bruins – Rutten 1961)
WO	Die Welt des Orients
ZA	Zeitschrift für Assyriologie und vorderasiatische Archäologie
ZDMG	Zeitschrift der Deutschen Morgenländischen Gesellschaft

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I. The starting point: Numbers or lines—in method and in conceptualization

For almost 60 years it has been known that the Babylonians of the Old Babylonian period[1] (and later) knew and solved equations of the second degree[2]—like this[3]

Obv. II, 1.	Length and width added is 14 and 48 the surface.	$x+y=14$ $x \cdot y=48$
2.	The magnitudes are not known. 14 times 14 (is) 3′16°.[4] 48 times 4 (is) 3′12°.	$14 \cdot 14=196$ $48 \cdot 4=192$
3.	3′12° from 3′16° you substract, and 4 remain. What times what	$196-192=4$
4.	shall I take in order to (get) 4? 2 times 2 (is) 4. 2 from 14 you subtract, and 12 remain.	$\zeta^2=4 \rightarrow \zeta=2$ $14-2=12$
5.	12 times 30′ (is) 6. 6 is the width. To 2 you shall add 6, 8 is it. 8 is the length.	$12 \cdot {}^1/_2=6=y$ $2+6=8=x$

1 The Old Babylonian period spans the time from c. 2000 B.C. to 1600 B.C. (middle chronology). The mathematical texts dealt with in this paper belong (with the exception of the Seleucid text presented first) to the time from c. 1800 B.C. to c. 1600 B.C.

2 Anachronisms are lurking everywhere when one speaks of Babylonian mathematics in modern terms. The Babylonians did not classify their problems according to degree. They have related classifications, but the delimitations deviate somewhat from ours, and they have another basis. "Equations", on the other hand, is a fully adequate description even of the Old Babylonian pattern of thought, if only we remember that what is equated is not pure number but the entity and its measuring number: Combinations of unknown quantities equal given numbers or, in certain cases, other combinations of unknown quantities.

3 BM 34568 No 9 (BM 34568 refers to the museum signature, No. 9 to the number of the problem inside the tablet as numbered in the edition of the text). The text was published, transliterated, translated and discussed by O. Neugebauer in MKT III 15ff. The numbers in the margin refer to the position of the text on the tablet: Obserse/reverse, column No, line No. The text is Seleucid, i.e. from around the 3rd century B.C. The translation is a literal retranslation of O. Neugebauer's German translation as given in MKT III. So, it renders the way in which Babylonian algebra is known to broader circles of historians of mathematics.—All translations given below will be my own direct translations from the original language.

4 For the transcription of the sexagesimal place value numbers found in the text I follow F. Thureau-Dangin's system, which in my opinion is better suited than O. Neugebauer's for the purpose of the present investigation: 3° is the same as 3, 3′ the same as $3 \cdot 60^{-1}$, 3″ means $3 \cdot 60^{-2}$, etc. 3′ means $3 \cdot 60^1$, 3″ equals $3 \cdot 60^2$, etc. The notation is an extension of our current degree-minute-second-notation, which anyhow descends directly from the Babylonian place value system.—I use the notation as a compromise between two requirements: For the convenience of the reader, the translations must indicate absolute place; this is not done in the original cuneiform, but so few errors are made during additive operations that the Babylonians must have possessed some means to keep track of orders of magnitude. On the other hand, the zeroes necessary in the conventional transcription introduced by O. Neugebauer (1932) (3,0;5 instead of F. Thureau-Dangin's 3\5′ and the Babylonian 3 5) are best avoided in an investigation of Babylonian patterns of thought, where such zeroes had no existence. Admittedly, the situation is quite different in an investigation of mathematical techniques, especially the techniques of mathematical astronomy, with special regard to which O. Neugebauer introduced his notation.

This short text will serve to locate the central question of the present paper. Apart from the statements of the problem and of the result, the text contains nothing but the description of a series of numerical computations – it can be characterized as an exemplification of an algorithm. Even problems 18 and 19 of the same tablet (MKT III,16f.), which describe a procedure abstractly, do so on the purely algorithmic level: "Take length, width and diagonal times length, width and diagonal. Take the surface times 2. Subtract the product from the ⟨square on length, width and⟩ diagonal. Take the remainder times one half . . .". There are no explanations of the way the solution is found, no justification of the steps which are made and, so it seems, no indication whatever of the pattern of thought behind the method.

Now it is an old observation that traditional algebraic problems can be solved by basically different (though often homomorphic) methods. So, if we look at a problem of the type $x+y=a$, $x \cdot y=b$, we would of course solve it by manipulation of symbols. Most Latin and Arabic algebras of the Middle Ages, from al-Khwārizmī onwards, would formulate it that "I have divided 10 into two parts, and multiplying one of these by the other, the result was 21"[5]; in order to obtain the solution, they would call one of the numbers "a thing" and the other "10 minus a thing", and by verbal argument ("rhetorical algebra") they would transform it into the standard problem "10 things are equal to 10 dirhems and a square", the solution of which was known from a standard algorithm. Diophantos would speak more abstractly of "finding two numbers so that their sum and product make given numbers"[6]; he would exemplify the method in a concrete case, "their sum makes 20 units, while their product makes 96 units", and he would proceed until the complete solution by purely rhetorical methods, formulated however by means of a set of standardized abbreviations ("syncopated algebra"[7]).

In the so-called "geometric algebra" of the Greeks, geometrical problems of the same structure are solved.[8] So, in Euclid's *Data*, prop. 85 it is demonstrated by stringent geometrical construction that "if two lines contain a given surface in a given angle, and their sum is also given, then they must both be given".[9]

Quite different geometry is used by al-Khwārizmī to justify the standard algorithms by means of which he solves the basic mixed second-degree equations. To avoid any confusion with the much-discussed "geometrical algebra" I will propose the term "naive geometry".[10] Since this concept will be fundamental for the following, I shall present it more fully.

5 Al-Khwārizmī, *Algebra*, tr. Rosen 1831: 41.

6 *Arithmetica* I, xxvii.

7 The term is due to Nesselmann (1842: 302ff.), who also introduced the more current "rhetorical algebra".

8 Irrespective of the question whether "geometric algebra" was or was intended to be an "algebra".

9 Cf. also *Elements* II 5. An analogue of the corresponding algebraic problem in one unknown is found in *Data*, prop. 58, and in *Elements* VI 28.

10 In a preliminary discussion paper (Høyrup 1985) I spoke of "geometrical heuristics". I have also pondered "visual" or "intuitive geometry". After much reflection, however, I have come to prefer "naive geometry" as relatively unloaded with psychological and philosophical connotations.

In order to justify his solution to the equation "square and roots equal to number", al-Khwārizmī explains the case "a Square and ten Roots are equal to thirty-nine Dirhems".[11] The number 39 is represented by a composite figure: A square of side equal to the unknown "Root" and two rectangles of length 5 ($=10 \div 5$) and width equal to the Root, positioned as shown in Fig. 1 (full-drawn line). The gnomonic figure is completed by addition of a square equal to $5^2=25$ (dotted line), the whole being then a square of area $39+25=64$. Its side being $\sqrt{64}=8$, the unknown Root will be $8-5=3$.

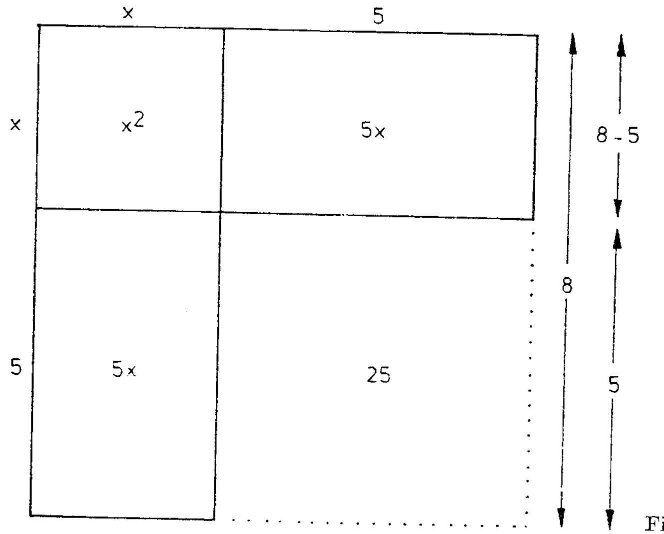

Fig. 1

We may feel comfortably sure that the argument behind our Babylonian algorithm was not of the Euclidean brand–Babylonian geometric texts show no trace at all of Euclidean argumentation. We can also safely exclude the hypothesis that the Babylonians made use of symbolic algebra.[12] Finally, we can

[11] See his *Algebra*, tr. Rosen 1831: 13–16.

[12] The immediate argument for this is that symbolic algebra requires a level of abstraction which appears to be totally alien to Babylonian thought. If this seems too much of an argument *ex silentio*, it can be added that symbolic algebra is *grosso modo* akin in structure to arithmetico-thetorical algebra. So, even if we upkeep the possibility of symbolic algebra as a silent hypothesis, the arguments which will be given later against an arithmetico-rhetorical interpretation will also exclude symbolic translations of the latter.–On the same account, an "abacus" representation of Babylonian algebra with counters representing the coefficients of the products and powers of the unknowns can be discarded. In itself, the "abacus interpretation" might have a certain plausibility, since material calculi had been used for common reckoning and/or computation in earlier epochs in Mesopotamia. Nothing, however, but the writing material, pebbles instead of ink, distinguishes such a representation from the syncopated algebra of Diophantos or the further development and schematization of the same principle found in Medieval Indian algebra. Arguments against an arithmetico-rhetorical interpretation of Babylonian algebra will hence also be arguments against an arithmetical "abacus algebra".–I shall return below to the possibility of a geometric "abacus algebra" related to the Greek "figurate numbers".

also be confident that some kind of argument lays behind the text. Random play with numbers might of course lead to the discovery of a correct algorithm for a single type of equation, and such an algorithm could then be transmitted mechanically. Still, the equation-types of Babylonian mathematics are so numerous, and the methods used to solve them so freely varied that random discovery cannot explain them. Some mental (and perhaps also physical) representation must have been at hand which could give a meaning to the many intermediate numbers of our algorithm (196, 4, 192, 4, 2, 12, $1/$) and to the operations to which they are submitted.

We cannot, however, read out of the text whether this representation was of rhetorico-arithmetical character or should be described as naive geometry. Truly, the "length", "width" and "surface" might seem to suggest the latter possibility. But even Diophantos used a geometrical vocabulary ("square", "application") which was only meant to suggest the arithmetical relations involved. Similarly, the Arabic and Latin algebras of the Middle Ages would speak indifferently of a second power as "square" or "property" and of a first power as "thing" or "root", intending nothing but suggestive words which might fill the adequate places in the sentences. So, no conclusion is possible on that level.

The procedure leaves us in no better situation. It is easy to devise a rhetorical method which yields the numbers of the text as intermediate results, viz. a verbal translation of this:

$$x+y=14; \qquad xy=48.$$
$$(x+y)^2=196; \qquad 4xy=192$$
$$(x-y)^2=(x+y)^2-4xy=196-192=4$$
$$x-y=\sqrt{4}=2$$

(the length is normally supposed to exceed the width; hence, no double solution will arise)

$$2y=(x+y)-(x-y)=14-2=12$$
$$y=1/_2 \cdot 12=6$$
$$x=(x-y)+y=2+6=8$$

It is, however, just as easy to devise a geometrical figure on which the correctness of the solution and of the single steps can be argued naively (see Fig. 2). Here, a geometrical counterpart of every single number occurring in the calculation can be found. So, the algorithm leaves us in a dead end: It fits equally well to a rhetorical argument by arithmetical relations and to an argument by naive geometry.

Concerning another aspect of the question arithmetic/naive geometry we are no better off than in the case of the method, namely regarding the conceptualization of the problem itself: Was it seen as a problem of unknown numbers, represented perhaps by the dimensions of a geometric figure, or shall it be taken at its words, as a problem really concerned with unknown dimensions of such a figure?

That this latter question must be separated from that of the character of the method can be seen from comparison with other algebraic traditions. It is clear that Modern mathematics thinks of a set of equations like $x+y=14$; $x \cdot y=48$ as concerned with numbers, and that we understand the operations used to solve it as purely arithmetical operations. So, the basis of Modern algebra is

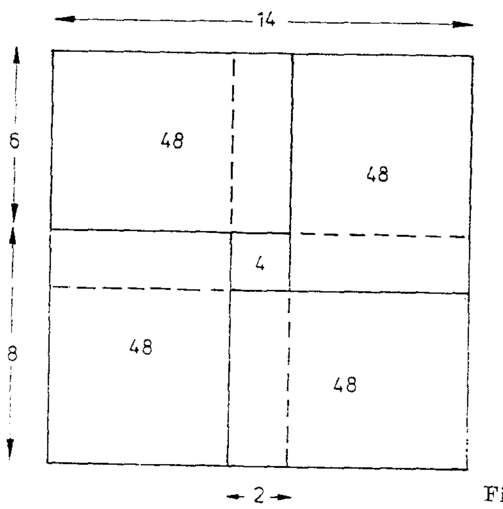

Fig. 2

arithmetical in conceptualization as well as method.[13] It is equally clear that we meet with lots of concrete problems, e.g. concerned with spatial extensions, which we translate into algebra and then solve by algebraic methods. In such cases, our conceptualization is concrete, e.g. geometrical, but our method is arithmetical—concrete entities are represented by abstract numbers.

On the whole, the same description would fit the Medieval algebraic tradition, with one important exception: The al-Khwārizmīan justification of the solution to the mixed second-degree equations (cf. above). There, the conceptualization of the problems is as arithmetical as everywhere else in al-Khwārizmī's algebra, but the method is naive geometry, where lines and surfaces represent the abstract numbers. Basic conceptualization and method need not coincide.

To anybody reading Babylonian "algebraic" sources it will be obvious that the conceptualizations of the problems are as varied as those of Modern algebra. Some are quite concrete geometrical problems: Partitions of triangular or quadrangular fields, calculations of the volumes of siege ramps, etc.; some are formulated as pure number problems, concerned e.g. with a pair of numbers belonging together in a table of reciprocals. The main body of texts, finally, deal with "lengths", "widths" and "surfaces" which cannot *a priori* be interpreted at face value, nor however as arithmetical dummies. Anyhow, there can be no reasonable doubt that these latter problems represent the basic conceptualization of Babylonian algebra, and that their "lengths" etc. are the entities which represent real lengths as well as numbers when such magnitudes occur in other problems.

[13] By "Modern" I mean "post-Renaissance", in the case of algebra specifically "post-Vieta". I disregard what mathematicians would call "modern" (abstract, "post-Noether") algebra as irrelevant to the present discussion: It is, at least in classical senses of these words, neither arithmetical nor geometric, be it in basic conceptualization or in method, although it is, primarily, an abstract extrapolation from arithmetical conceptualization and method.

There are, then, two main aspects of the problem investigated below: Firstly, whether the method used in Old Babylonian algebra was arithmetical (rhetorical or related) or naive-geometrical. Secondly, whether its basic conceptualization was arithmetical or geometrical.[14] Around these basic questions a web of other related and derived discussions will be spun, in order to give an allround picture of the discipline.

II. The obstacles

Neither the terminology nor the procedure of the problem translated above, would permit us to decide this question, or just to approach it. In this respect it is similar to a great many other Babylonian texts. For half a century, in has therefore been the prevailing opinion among historians of mathematics that at least the surviving and published texts will not permit us to solve the dilemma arithmetic/geometry. At the same time, most historians have implicitly or explicitly tended to favour the fully arithmetical hypothesis[15]—with the partial exception of K. Vogel, A. A. Vajman and B. L. van der Waerden.[16]

Until the Summer 1982, I shared these common opinions and prejudices, as I would now call them. At that time, however, I was inspired, by an interpretation of a puzzling text[17] and by a critical question from P. Damerow for my reasons, to look for traces of geometrical thought in other texts. Since then my knowledge of the language has improved so much that I have come to regard my original textual inspiration as totally wrong.[18] But like another Columbus I had the good luck to hit land on a course which I had chosen for bad reasons. A close reading of the texts, and the use of methods closer to those of contemporary human sciences (linguistics and structural semantics as well as literary analysis) than to those traditionally used in the history of Ancient mathematics, revealed that the arithmetical hypothesis cannot be upheld. As it is always more

[14] It should be emphasized that the investigation deals only with the algebraic texts. There is no reason to doubt the purely numerical character of many of the table texts; but the numerical character of texts like Plimpton 322 (MCT 38) does not permit us to conclude that algebraic problems, too, were understood and solved arithmetically. Similarly, it cannot be doubted that a number of texts deal with real geometric problems,—but even there generalizations are not automatically justified.

[15] Among the most explicit, Thureau-Dangin (1940: 302) states that the problems dealing with geometrical figures do so because "a plane figure will easily give rise to a second-degree equation", but that the problems are still "purely numerical", just like the indeterminate equations of Diophantos' *Arithmetica* VI, for which right triangles function merely as a pretext.

[16] So, van der Waerden (1961: 71f.) suggests hypothetically that certain basic algebraic identities may have been proved geometrically ($\{a-b\}\{a+b\}=a^2-b^2$, etc.). The conjecture is accepted by Vajman (1961: 168f.). At the same time, however, B. L. van der Waerden distinguishes the method of proof from the conceptualization, stating that the "thought processes of the Babylonians were chiefly algebraic (i.e. arithmetico-algebraic—J. H.). It is true that they illustrated unknown numbers by means of lines and areas, but they always remained numbers".

[17] IM 52301, the inscription on the edge as interpreted by Bruins (1953: 242f., 252).

[18] Cf. the revised transliteration and the new discussion of IM 52301 in Gundlach – von Soden 1963: 253, 259f.

difficult to verify than to falsify, I cannot claim that the investigation has proved a specific geometrical interpretation to be correct. Still, the geometrical reading gets very strong support, and I think it can be taken for sure that the Old Babylonian algebra must at least have been structurally isomorphic to a representation by naive geometry, while the arithmetical representation is only a homomorphism.

It will be clear from the following that my results could not have been found without methodological innovations. So, we should not wonder that the evidence against arithmetical thought has gone largely unnoticed for 50 years, and that the interpretation which O. Neugebauer characterized as a "first approximation" in 1932[19] has stood unchallenged since then.

This may sound cryptic to readers who are not familiar with the cuneiform script and texts, and may require an explanation. The Babylonian texts were written in a Semitic language (Akkadian) which has been dead as a literary language for two millennia (and as a spoken language even longer), with strong, at times all-dominating admixtures of loanwords from another language (Sumerian), which was probably already dead around c. 1800 B.C. except as a literary language used by the restricted circle of scribes, and of which no relative is known. Even the interpretation of the Akkadian language is far from completed, and the situation for Sumerian is still worse.[20] To add to the confusion, the script used consists of signs which may stand for one or, normally, several phonetic values, not necessarily close to one another, and for one or often several semantic ("ideographic") values, i.e. values as word signs ("logograms")[21] for Sumerian words and semantically related Akkadian words. The connection between the different values is rooted in semantic affinity, in phonetic affinity in either of the two languages, or simply in the conflation of originally separate signs.[22] To all this may come trite problems of legibility, due to careless writing or to bad preservation of the tablets.

[19] Neugebauer 1932a: 6.

[20] So, no real Sumerian dictionary exists to this day.

[21] The prevailing tendency has been to leave the conception of ideograms and to claim that the cuneiform signs when not used phonetically would stand for, and be read as, specific Akkadian words. The difference between an ideogram and a logogram is as the difference between "+" and "viz.": The first sign will of course always be read by words, depending on the situation as "plus", "added to", "and", or something similar; only in the specific additive meaning, however, can it replace the spoken word "and"; it is no logogram, it corresponds to an operational concept which is not identical with any verbal description. "Viz.", on the other hand, is a real logogram for "namely".—No doubt, the logographic interpretation describes the normal non-phonetic use of cuneiform signs adequately. At least in mathematical texts, however, certain signs must be understood as ideograms, not as logograms, as I shall exemplify below (cf. notes 57f. and note d to TMS XVI A; cf. also SLa 25f., on similar phenomena in non-mathematical contexts).

[22] The sign may be taken as an example. The conventional sign name is KAŠ, the name given to it in ancient sign lists. It may stand for Sumerian k a š, "beer" (Sumerian words are usually transliterated in spaced types), and for the possessive suffix - b i ; the latter reading is used in Sumerian as an approximate syllabic writing for the compound b + e > b é, "says it" (or rather "it is said"). These three uses have given rise, respectively, to logographic use in Akkadian texts for the corresponding words *šikarum*, *-šu/-ša* and *qabûm*, together with the derived *šū/šūatu*, "this", a function in which Su-

Happily, the system was also ambiguous for the Babylonian scribes themselves, and they developed certain aids for avoiding the ambiguities (phonetic complements to logograms; semantic determinatives). Furthermore, inside texts belonging to a specific type and period, the range of possible values of a given sign is strongly restricted. The restrictions, however, have to be discovered; hence, extensive knowledge of a whole text-type is required before the single text can be safely transliterated into syllabic Latin writing.

On this background, the immensity of the task solved in the 1930es by O. Neugebauer and F. Thureau-Dangin will be seen: To decipher the phrasing of the mathematical texts, and to discover the mathematical meaning of the terms. First when this is done in a way which can be relied upon can the question of conceptualization be raised in earnest.

Raised . . . but hardly solved by direct methods. Just because the language of the single text-type is specific, we must regard the terminology as technical or semi-technical. We know from modern languages that the semantic contents of a technical term are not necessarily unravelled by etymological studies. The etymology of "perpendicular" would lead us to the pending plumb-line and thus to the vertical direction. *A posteriori* we can understand the way from here to the right angle—but we cannot predict *a priori* that "vertical" will change into "right angle", nor can we even be sure that a modern geometer thinks of verticality when he uses the standard-phrase and raises a perpendicular.[23]

The situation is not very different in Akkadian, or in Semitic languages in general. An example from the Hebrew on which I shall draw below will show this. *'bq* has, as a verb, the meaning "to fly away". Hence we have nominal derivations "(light) dust" and "pollen" (HAHw, 7[a]); from "light dust" probably the tablet covered with light dust or sand, the "dust abacus", and from here apparently the "abacus" in general.[24] Who would imagine that the heavy table on which stone calculi are moved was, etymologically, "something flying away"?

Truly, the character of Semitic languages is such that the basic semantic implications of the root from which a word derives are rarely or never lost quite of sight—they are conserved at least as connotations. Such conservations are forced upon the users of the language by its very structure.[25] But a requirement

merian b i can also be used. In the Old Babylonian period it will also be found with the phonetic values *bi*, *bé*, *pí*, and *pé* (accents and subscript numbers are used to distinguish different writings of the same syllable). In later periods, it can also be used phonetically as *gaš*, *kaš* and *kás*.—To this comes the role in a number of composite sign groups used logographically: different specified sorts of beer; innkeeper; song(?); etc. Finally, the sign may represent twice the surface unit è š e, written . (After MEA and ABZ No 214, and a commentary from B. Alster).

[23] To know whether he thinks concretely through the standard-term we would have to investigate whether he avoids using it when constructing the orthogonal to a non-horizontal line; i.e., we would have to investigate the structure of his total terminology and its use in various situations.

[24] See Pauly-Wissowa I(i), 5. HAHw quotes the Semitic root in Hebrew, Arabic and Aramaic. It appears to be absent in Akkadian.

[25] The Semitic languages combine—with special clarity and richness in the system of verbs and their derivations—fixed, mainly consonantal roots carrying the semantic basis, with a huge variety of prefixes, infixes (among which the vowels, which are sub-

that there should be a semantic umbilical chord between the general and the technical meaning of a term can at most be used as a control with hindsight, when the technical meaning has already been interpreted tentatively. It can tell nothing in advance.

In principle, technical terms should therefore be interpreted from technical texts. Here, more than anywhere else, the Wittgensteinian dictum should be remembered: "Don't ask for the meaning–ask for the use". Then, however, we are led into a vicious circle: Our sole access to the use of the technical terms is the body of texts, which only tell us about the use if we understand their terms. As long as two conflicting interpretations of the terminology both permit coherent understanding of use and meaning, neither can be rejected. And indeed, if we believe in an arithmetical interpretation of Babylonian algebra, we are led to an arithmetical interpretation of the unknown terms denoting its operations, and thus to a confirmation of our initial beliefs; initial belief in a geometrical interpretation is, however, equally selfconfirming.

Let us take an example, the phrase

10 *itti* 10 *šutākil-ma*: 1′40°.[26]

itti can be translated "together with", and the enclitic particle *-ma* by "and then" or "and thus", or it can simply be represented (as I shall do in the following) by ":". So, the phrase can be partially translated as

10 *šutākil* together with 10: 1′40°,

and so we know that *šutākil* represents an operation which from 10 and 10 creates 1′40° (=100), either an arithmetical multiplication of pure numbers, or a geometrical operation creating a rectangle with sides 10 and 10 and a corresponding surface of 100. The form can also be recognized as the imperative of a reciprocative causative stem derived from *akālum*, "to eat", or from *kullum* "to hold" (in which case the transcription ought to be *šutakīl*).[27] Hence we have the interpretation

"Make 10 and 10 eat/hold each other: 100,"

or, if we do not see what "eating" or "holding" has to do with the matter, and if

mitted to change) and suffixes determining not only grammatical category but also many semantic displacements which in Indo-European languages are not subject to morphological regularity. The actual functioning of such a system requires that its speakers apprehend subconsciously all the derivations of a root as belonging to one scheme, in the way an English four-year old child apprehends "whistled" as a temporal displacement of the semantic basis "whistle" according to a general scheme, as revealed by her construction of forms like "goed" instead of "went".

[26] VAT 8390 rev. 21 (MKT I 337).

[27] The former interpretation is suggested by the use of the Sumerian kú, "to eat", as a logogram for the term (cf. below section IV.2). For this reason it is normally accepted today, cf. von Soden 1964: 50, and AHw, *kullu(m)* and *akālu(m)*.–The latter interpretation was proposed by F. Thureau-Dangin (e.g. TMB 219), who explained the logographic use of kú as a pun-like transfer, inspired by coincident Št-forms for *kullum* and *akālum* (cf. TMB 232f.). Such transfers are in fact not uncommon in cuneiform writing (cf. above, note 22), and hence a derivation from "holding" cannot be outruled. –As it will appear below, a relation to another term *(takīltum)* appears to rule out the derivation from "eating", while a connection to "holding" makes perfect sense (cf. below, section IV.3). On the other hand, A. Westenholz expects that *kullum* would give rise to the form *šutkīl* and not to *šutakīl*–which I cannot make agree, however, with a number of derivations from *ḫiaqum*. Most safely, the question is left open.

we want to keep the question explicitly open, we may represent the semantic basis through a dummy XX:

"Make 10 and 10 XX each other: 100."

In both ways, we get something like idiomatic English as translation of the phrase. Still, concerning the question arithmetical versus geometrical interpretation we are no more wise.

Truly, most standard terms of Babylonian algebra look less opaque than "mutual eating/holding". "To append" x to y, "to pile up" x and y; "to tear out" or "to cut off" x from y or to see "how much y goes beyond x"; "to break x to two"; all of these can, as descriptions of additive and substractive procedures and of halving, respectively, be interpreted concretely, and all seem to suggest an imagination oriented toward something manifest, e.g. the procedures of naive geometry, rather than an abstract arithmetical understanding. But so do the Latin etymologies of "addition" and "subtraction"; like these, several of the Akkadian terms were established as standard expressions, and some may have been fixed translations of age-old terms. There may have been as little concrete substance left in them as there remains of lead in a right angle.

On the level of single terms and their applications the texts are thus not fit to elucidate the conceptual aspects of Babylonian algebra and mathematics.

III. The structural and discursive levels

Originally, I started my search for traces of naive-geometrical thought precisely at the level of single-term applications and literal meanings, and I was soon able to draw the negative conclusions just presented. At the same time, however, the close reading of the texts had led me to some real clues. One of these was the structure of the total mathematical terminology used in the Babylonian algebraic texts.[28] The other has to do with what could be called the "discursive aspect" of the texts (as opposed to technical and terminological aspects): The way things are spoken of and explained, the organization of explanations and directives, and metaphorical and other non-technical use of seemingly technical terms.[29]

[28] A simple instance of such structural analysis was suggested in note 23 as a means to investigate whether a modern user of geometrical terminology associates the "raising" of a perpendicular with the literal meaning of this term.

[29] This paradoxical phrase should perhaps be clarified. An important characteristic of a technical term is fixed semantic contents and relative absence of connotations and analogic meanings. Technical terms when applied as such are not open-ended. Even in modern mathematics, however, technical terms are also used metaphorically and in other ways departing from their technical semantics. This happens during theoretical innovation, when the technical terminology has to adapt to new conceptual structures. It also occurs in informal discussion and didactical explanation when truth is not to be stated but to be discovered or conveyed. These are processes which always require compromise with pre-existent understanding, and therefore such non-technical displacements of meaning reveal something about this understanding. (Cf. for certain aspects of this discussion Beck 1978 and Marcus 1980).

The Babylonian mathematical texts abound in examples of such derived meanings and applications of terms to an extent which suggests that we are not confronted with

The clues implied by the discursive aspect of the texts can only be demonstrated on specific examples, and I shall postpone their presentation. Part of the evidence provided by the structural analysis can, on the other hand, be explained in abstract form. Instead of retelling my Odyssey through the texts completely and from the beginning,[30] I shall therefore present some basic results abstractly before going on to a selection of texts in order to penetrate further. Exemplifications and supplementary arguments will be given on the basis of these texts.

In current English, the expressions "*a* times *b*" and "*a* multiplied by *b*" describe the same process–they are synonyms. Which one to choose in a given situation is a matter of style–as will be demonstrated by the fact that person A may choose the one in a situation where person B would choose the other, or that the choice depends on audience (school children versus mathematicians) or medium (oral or written, popular or scholarly). We have two different expressions at our disposal, but we have only one mathematical concept.

The Babylonians had many multiplicative expressions: *šutākulum* (whence *šutākil*); *našûm*; íl; nim; *eṣēpum*; tab; a-rá; UL.UL; UR.UR. The matter has, to my knowledge, never been discussed explicitly, but it has been taken for granted and selfevident that all[31] described the same concept.[32]

As long as an arithmetical conceptualization was itself taken for granted, and taken for granted to such an extent that the mere possibility of alternative conceptualizations was not recognized, this automatic conflation of all multiplicative concepts was unavoidable: In an arithmetical conceptualization there is only one operation to be described, there can be only one concept.[33]

Still, selfevident as it has appeared to be, the conflation is not true to Babylonian mathematical thought. The terms are not synonyms, the choice among them is restricted by other criteria than those of style, taste and dialect.

Truly, some sets of terms are synonyms. íl is the Sumerian equivalent of *našûm*, "to raise", and it is used logographically in exactly the same functions (which makes it debatable whether we are entitled to speak of a different term–*našûm* and íl are rather full and shorthand writings of the same Akkadian term). nim, Sumerian equivalent of *elûm*, "to be high" and used even for its factivitive stem "to elevate", is used instead in a few texts (here, then, another term for

a real technical terminology after all, that few terms possess a basic, really fixed technical meaning. Instead, most terms should probably be regarded as open-ended expressions which in certain standardized situations are used in a standardized way. This will be amply exemplified below.

30 This is, *grosso modo*, the way I go through the subject in my preliminary presentation (Høyrup 1985) of the problem and of my results. The outcome is rather opaque.

31 With the partial exception of *eṣēpum* and its logographic equivalent tab, the original meaning of which is "to duplicate", and which in phrases "duplicate x to n" means "multiply x by (the positive integer) n" if interpreted arithmetically.

32 It should, however, be emphasized that both O. Neugebauer and F. Thureau-Dangin show great intuitive sensitivity to the shades of the vocabulary in MKT and TMB. I remember no single restitution of a broken text in either of the two collections which does not fit the results of my structural investigation.

33 Disregarding the possibility to distinguish between multiplications involving only integers, multiplications where one factor at least is an integer, and multiplications of wider classes of numbers. In fact, all Babylonian terms except *eṣēpum* (and tab) can be applied for the "multiplication" of any number by any other number.

the same concept is in play–the equivalence is semantic but no longer logographic). Similarly, *šutākulum* (with the logographic writing kú) is replaced by UL.UL in certain texts and by UR.UR in others. But while the choice of a term inside a group is free, the choice of the group from which a term shall be taken is subject to clear rules–rules which in a geometrical interpretation of the procedures are easily stated.

IV. Basic vocabulary and translational principles

Most other classes of arithmetical operations are also subdivided in Old Babylonian mathematical thought, if we are to judge from the Old Babylonian vocabulary.[34] As a preparation for the presentation of the texts, I shall summarize in schematic form the basic vocabulary and its subdivisions, indicating in rough outline the use of each subclass. I shall also give the "standard translations" of the terms which I am going to use in my translations of texts in the following chapters, together with the translations of the terms given in AHw.[35]

IV. 1. Additive operations

Two different "additions" are distinguished. The first is described by the term *waṣābum* (AHw "hinzufügen"), and it is used when something is added to an entity the identity of which is conserved through the process (the nominal derivative *ṣibtum* designates *inter alia* the interest, which does not change the identity of the capital to which it is added). The Sumerian daḫ is used as a logogram. In order to avoid associations to the modern abstract concept of addition, I use the standard translation "to append" for both terms.

The other addition is designated by *kamārum* (AHw "schichten, häufen"). It is used when several entities are accumulated into one "heap" (cf. the etymology of "accumulation" from "cumulus"), which is identical with neither of them. gar-gar and UL.GAR are both used ideographically in the same function[36], apparently as pure logograms. For standard translations of all three terms I use "to accumulate".

While no separate name for the sum of an "identity-conserving" addition is found (for good reasons, of course), the "accumulation" can be designated by various derivations of *kamārum*: *kimrātum*, a feminine plural[37] (whence my

34 The vocabulary of the later (Seleucid) mathematical texts is very different, and can indeed be taken as an indication that the mathematical conceptualizations had changed through and through during the centuries which separate the two periods. Cf. below, section X.2.

35 In order to emphasize the purely Old Babylonian character of the summary I write all Akkadian verbs and nouns with "mimation", i.e. with the final *-m* which was lost in later centuries.

36 Literally, the Sumerian gar-gar means something like "to lay down (gar) repeatedly"; possibly, the UL of UL.GAR is due to a sound shift from UR = ur, *inter alia* "to collect" (ŠL II No 575.9), which would lead to an interpretation of UL.GAR as a composite verb "to lay down collectedly" (maybe an artificial "pseudo-Sumerogram").

37 Cf. section VIII.2, the notes to AO 8862, for reasons why the single sum has to be understood as a plural.

standard-translation "things accumulated"), *nakmartum* (standard translation "accumulated") and *kumurrûm* ("accumulation"). gar-gar and UL.GAR can both serve logographically in the same functions.

IV. 2. Subtractive operations

Subtractions too may and may not conserve identity. The "non-conserving" subtraction compares two different entities, by means of the expression *mala x eli y itter*, "as much as *x* over *y* goes beyond" (from *watārum*, "übergroß, überschüssig sein/werden", with the logograms SI and dirig). The most common term for the "identity-conserving" subtraction is *nasāḫum*, "ausreißen", with logographic equivalent zi. I shall use the standard translation "tear out". Another term with the same function (but apparently a slightly different shade) is *ḫarāṣum*, "abschneiden" (etc.), st. transl. "cut off". In specific situations, a variety of other terms may occur.

IV. 3. Multiplicative operations

The standard expression of the multiplication tables is "*x* a-rá *y*" where *x* and *y* are pure numbers. It is also found in a few of the problem texts (normally in double constructions, cf. below). The semantic base is rá, "to go" (cf. Danish *gange*, "times", from *gå*, "to go", and the analogous Swedish terms). After having used initially the modernizing standard translation "*x* times *y*" for "*x* a-rá *y*" I have opted for "*x* steps of *y*", mainly because even Seleucid texts remember this sense of the term, as revealed by their use of a genitive for the second factor (cf. below, section X.2, BM 34568 No 9; cf. also note 38).

The term *eṣēpum* (AHw "verdoppeln") and its equivalent tab "to duplicate", i.e. "to take once more", whence even the extension "to repeat several times", was already mentioned. It is used for multiplications of any concrete entity by a positive and not too large integer, and apparently meant as a concrete repetition of that entity. When used to "make multiple", it occurs in phrases like "*X ana n eṣēpum*", "to repeat *x* until *n*", or "*x* a-rá *n* tab", "to repeat *x n* steps" (the deviating use of a-rá will be noticed[38]). In all cases, I use the standard translation "to repeat".

The third group is made up of *našûm* ("(hoch)heben, tragen"), its Sumerian equivalent íl (the normal logogram for *našûm*), and the Sumerian nim, apparently also used logographically in certain texts. As mentioned above, the latter term means originally "be high", equivalent of Akkadian *elûm*. In mathematical contexts it is in all probability used as a pseudo-Sumerogram for the (factitive) D-stem *ullûm* of this word.[39] These terms are used for the normal calculation of concrete quantities by multiplication: When multiplying by the tabulated constant (igi-gub) factors; when multiplying by a reciprocal as a substitute for division (cf. below); in all situations involving a factor of proportionality; and when the areas of trapeziums, triangles and trapezoids are found.[40] As standard translations I use "to raise" for *našûm* and íl (the alternative "to carry" cannot be brought into semantic harmony with nim). For nim I use "to lift".

[38] A similar use of Akkadian *alākum*, "to go", as a substitute for *eṣēpum* is found in several Susa texts (among which TMS IX, translated below in section VIII.3). In one of them (viz. TMS VII) the "step" which is gone repeatedly appears to be designated a-rá.

[39] Originally, Thureau-Dangin suggested the conjecture that nim might be used for the factitive or causative Š-stem *šūlûm* (TMB 239). However, the headline of the Susa list

The connection between "raising" and multiplication is not obvious to the modern mind. Several clues exist in the texts, however, which connect the usage to Babylonian technical practice.

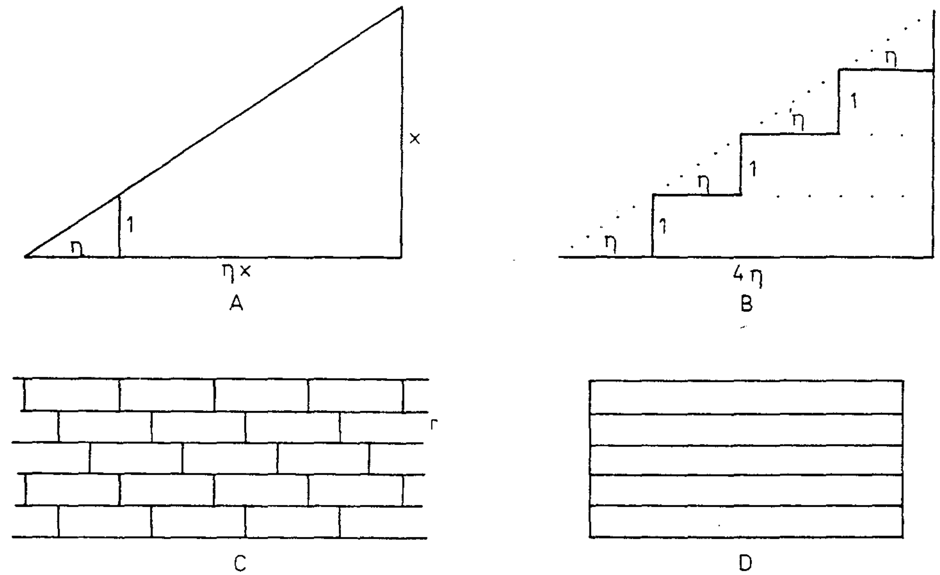

Fig. 3

One clue derives from the way volumes are calculated. If the base is quadratic, rectangular or circular, it is normally "spanned" by length and width (or found as $^1/_{12}$ of the area spanned by the circular circumference with itself). The multiplication with the vertical dimension, however, is a "raising" or "lifting". In itself, this already speaks to the imagination—raising *is* vertical movement. Furthermore, *ullûm* (and hence nim, cf. above) is precisely the term used when a wall is elevated *n* brick layers (AHW 208 b$^{11-14}$).

Another clue is provided by the use of the expression "íl^{tum} of 1 cubit (height)" (íl with phonetic complement *-tum*, indicating a derivation from *našûm* with ending *-tum*, e.g. *našītum*, a substantivized participle meaning "that which raises") as a measure for the

of constant (igi-gub-) factors claims to contain "igi-gub, that of making anything high" (TMS III 1), using the infinitive *ullûm* of the constantly factitive D-stem. Since the Š-stem is furthermore used (in AO 17264, MKT I 126f., and in Haddad 104,III,7, al-Rawi—Roaf 1984) in the sense of making a square-root "come up" as a result, nim ~*ullûm* is probably to replace F. Thureau-Dangin's conjecture.

[40] As we shall see below, the area of a rectangle is presumably also found by "raising", although the operation is normally not made explicit.

inverse gradient of a slope, i. e., the length one has to progress horizontally in order to attain an elevation of 1 cubit.[41]

Fig. 3A shows the situation, demonstrating the role of the íl^{tum} (η) both as a factor of proportionality and as "that which raises the slope 1 cubit". Fig. 3B shows the same in a less sophisticated manner (for which reason it is used occasionally in modern elementary teaching), closer to the Babylonian term than the Greek-type Figure 3A.

Comparison of Fig. 3B and Fig. 3C shows that the "raising of a slope" and the "raising of a wall" can easily be imagined as the same process. Figure 3D, finally, dem onstrates how the conception of a rectangular area as consisting of unit strips which is testified by the terminology (cf. below, section VII.2) can make one assimilate even area calculation to the same scheme.

Sargonic and earlier mathematical texts contain many area computations but never any term for multiplication. Brickwork and slope calculations seem to have arisen later—the oldest mathematical brick text known is from Ur III.[41a] We may imagine that explicit multiplicatory terminology was introduced together with these "new multiplications", and that it was then also used metaphorically for other similar calculations, be it area computations or arguments of proportionality. In this connection one should remember that not only the use of igi-gub-factors but also the computation of a/b by means of a table of reciprocals (cf. section IV.6) builds on proportionality.

The last group of multiplicatory operations is made up by *šutākulum*, "to make eat/hold each other", and its various semantic cognates: ì-kú-kú and ì-kú (its logograms), UL.UL and UR.UR. Some further cognates turn up below under the heading "squaring". In the algebra-texts, these terms are only used when an entity which may be considered a "length" is multiplied by another which is a "width", or by itself. That is, in a geometric interpretation of the texts it is used when a rectangle or a square is considered, in fact, as we shall see below, when it is produced. To a modern mind it might be tempting to interpret this as an indication that the term is used for the calculation of an area, since this involves the multiplication of two quantities of dimension length. The falseness of such an interpretation is, however, obvious from the way the areas of triangles, trapeziums and trapezoids are found: As soon as calculated average lengths are multiplied, the term used is *našûm*, íl or nim.

The interpretation of *šutākulum* understood as "mutual eating" is less than self-evident. Truly, an idea which was advanced by S. Gandz[42] in order to explain the use of *ukullûm*, "ration of food", as a term for the inverse gradient of a slope, could be extended as a last resort: In Hebrew, a field covered by vines is said to be "eaten" by the vines.[43] Similarly, a "mutual eating" inherent in *šutākulum* could be read as "mutual covering". To "make length and width cover each other" should then mean "to make them define/confine" a surface—viz. a rectangular surface, since it is fully described by length and width. The case where "length and length" are made eat/hold/cover each other,[44] on the other hand, turns out to describe the construction of an irregular quadrangle.

[41] BM 85196 rev. II 11 (MKT II 46).
[41a] N.C. 304, see Vajman 1961: 246f., cf. for the dating Friberg (forthcoming) § 4.5.
[42] Gandz 1939: 417f.
[43] The same idea of covering a piece of land is indeed seen in the Old Babylonian measurement of a slope by the "*ukullûm* eaten in 1 cubit", i.e. covered per cubit height (VAT 6598 rev. I 18, in MKT I 279, cf. TMB 129).
[44] YBC 4675 obv. 1 (MCT 44) has the expression *šumma* a. šà uš uš i-kú", "when a length and a length eat/hold a surface", referring to a surface stretched by two (dif-

It would, however, seem much more obvious to conceptualize the situation as a length and a width (or a length and another length) "holding" together the rectangle (or trapezoid) in question. In either case the geometrical contents of the metaphor is the same, the two lines confining together a surface. As standard translation I shall use the phrase "make A and B span" (which should be neutral with regard to the two possible derivations though slanted towards "holding"). Two texts (VAT 8390 and AO 8862, cf. below) make explicit that surface construction is meant, telling that "length and width I have made span: A surface I have built".

The ideogram ì-kú-kú seems to derive simply from the reciprocity of the Št-stem (the form ì-kú being a mere abbreviation: it is mainly used in the utterly compact "series texts"). UR.UR and UL.UL have the same repetitive structure; their semantics is probably best explained in connection with the concepts for squaring, to which we shall turn next.

As it will be seen below, the term *takīltum* (read as *šakīltum* in MKT I), which turns up in specific connections during the solution of second-degree-equations, must be related to *šutākulum*; I shall use the term untranslated. Detailed discussions of its meaning and use must await its occurrence in the texts. At present it should only be observed that according to all available evidence it cannot derive from *akālum*, which forms no D-stem. Its close connection to *šutākulum* implies that the derivation same must hold for the latter term (in which case, by the way, the correct transcription will be *šutakūl(l)um*), cf. note 27).

IV. 4. Squaring and square-root

The two fundamental verbs belonging to this area are si$_8$, "to be equal", and *maḫārum*, "gegenübertreten (as an adversary, as an equivalent)" etc. From the mid-third millennium onwards, si$_8$ is used to denote a square as (a quadrangular figure with) equal sides. At approximately the same early epoch, it is also seen to denote the equality of the lengths alone or the widths alone in quadrangles.[45] In the Old Babylonian texts, it is found with a prefix as íb-si.[46], literally a verbal form, probably meaning "it makes equal". It is used when square-roots are extracted, at times inside constructions where it stands clearly as a verb, at times seemingly as a noun identifying the square-root itself. In YBC 6504 (MKT III 22f.) and in the "series texts" it is used for (geometrical or arithmetical) squaring (cf. note 63), and in one text[47] it denotes an indubitable geometric square.

ferent) lengths, i.e. to an irregular quadrangular surface. Later in the same text (rev. 15) the term *šutākulum* itself stands as a complete parallel to the use (in rev. 6) of *epēšum*, "to make", "to produce" (viz. a quadrangular surface). In neither case is any multiplication to be found.

[45] On the denotation of squares, see Deimel 1923 No 82 (cf. MKT I 91, and Powell 1976: 430) and Edzard 1969. On the equality of lengths alone or widths alone, see Allotte de la Fuÿe 1915: 137ff.
[46] Occasionally ba-si$_8$. This term is, however, more common in connection with cube roots.
[47] BM 15285, passim (MKT I 137f.). The geometrical character of the squares is certain

To a modernizing mathematical interpretation this looks like primitive confusion: The Babylonians use the same term for a square (number) and its square root. Such a reading is, however, anachronistic, due to a pattern of thought which would have looked confused to a Babylonian: We conflate the geometrical figure characterized by equal and mutually orthogonal sides with one of its attributes, viz. the area which can be ascribed to it (the square "is" 25 m^2, while it "has" a side of 5 m). The Babylonians conflate the figure with another attribute, viz. with its side (the square figure "is" 10 nindan, while it "has" an area of 1 iku=100 nindan^2). Following a proposal by J. Friberg[48], I shall use the standard translation "equilateral" in cases where the term is used as a noun. This should avoid the wrong connotations following from the use of words bound up with our own conceptual distinctions and conflations. When the term is used as a verb, I shall use "to make equilateral"–the reasons for this will be given below on the basis of the texts.

maḫārum itself is mostly used in mathematical texts in the sense of "correspond to/confront (as equal)" ("confront" will be my standard translation). The derivation *mitḫartum* (a nominal derivation, "thing characterized by correspondence/counterposition") is used to denote a square, i.e., as we shall see in the following chapter, a geometrical square–once again identified with its side and possessing an area.[49] I shall use the standard translation "confrontation", in agreement with a conception of the square as a "situation" determined by confronting equals. The verbal Št-stem *šutamḫurum* ("to make correspond to/make confront each other") is used for the process of squaring with only one number or length as the object. I shall use the standard translation "make confront itself", viz. so that a square is formed.

A final important derivative is *meḫrum* (for which gaba(-ri) appears to be used logographically), "that which corresponds to/confronts its equal". Its function is best explained in connection with occurrences in the texts, so I shall postpone it. As standard translation I use "counterpart".

A number of other terms and signs belong to the same semantic field. LAGAB (written KIL in MKT and TMS) is used in one text[50] to indicate equality between shares in a field partition; in the "Tell Ḫarmal compendium"[51] and in one of the Susa texts[52] it denotes the usual square figure ("being" a length and "possessing" an area. Basing myself on the Tell Ḫarmal compendium I shall treat it as a logogram for *mitḫartum*, giving it the same standard translation.[53] NIGIN (written KIL.KIL in MKT) is used in one

both because they are spoken of as positioned and because they are drawn on the tablet. Shifts between the two terms show that íb-si$_8$ is intended here as a logogram for the Akkadian word *mitḫartum* (cf. immediately below). In the "algebraic" problem text Str. 363 (MKT I 244), where the scribe has done his best to find (and, one may suspect, to construct) Sumerian logograms to express his Akkadian thought, the same equivalence íb-si$_8$ ~ *mitḫartum* is used.

48 Private communication.

49 See e.g. BM 13901 passim (several problems are translated below).

50 AO 17 264 obv. 2f. (MKT I 126).

51 Goetze 1951.

52 Texte V, TMS 35ff. All three occurrences are late Old Babylonian, AO 17 264 possibly even early Kassite.

53 The sign is indeed a "confrontation" of equal lines: . It is thus probable that its ideographic equivalence with *mitḫartum*, rather than being connected to its use as a

Susa text[54] exactly as LAGAB, for the square figure. In the larger part of the Susa corpus it could be replaced by *šutamḫurum*, as also in some genuine Babylonian texts.[55] Finally it is found in a couple of Susa texts with two factors[56], corresponding to the use of *šutākulum*. This practical equivalence with several semantically related yet glossarially distinct terms makes it impossible to consider it a real logogram for any of its equivalences; hence, NIGIN is an example of a non-logographic ideogram.[57] Since the sign can replace *lawûm*, "umgeben", *saḫārum*, "sich wenden", "herumgehen" and its derivative *siḫirtum*, „Umkreis", I shall propose the standard translation "make surround", viz. surround a square or rectangular figure, and square or rectangular "surrounding", depending on the word class required by context.

UR.UR is found in certain texts in constructions similar to those with *šutākulum*.[58] UR itself is found in another late Old Babylonian or early Kassite text[59] in the sense of "squaring", and in general non-mathematical language it can be used logographically (with various complements) for *ištēniš*, "like one", "together" (<*ištēnum* "one"), for *mitḫāriš* "correspondingly" (i.e. "equally" or "simultanously", <*maḫārum*, cf. above),

logogramm for *lawûm*, "to surround" (in which case its Sumerian reading is nigín), is to be considered directly iconic.–In any case, the use of the sign in AO 17264 (cf. note 50) must be considered secondary, derived from the habitual association of the quadratic figure with equality. In this connection it is perhaps worthwhile remembering that the sign for si$_8$ was also originally (and still in Old Babylonian inscriptions on stone) a square standing on a corner (and , respectively). Even this sign would thus have directly iconic connotations.–It should be observed that the evidence for logographic equivalence from the Tell Ḫarmal compendium is evidence for the way it was read aloud but not necessarily for complete identity (nowadays, "+" may be read aloud as "and", but the context will show that addition is meant). Precisely this text, indeed, contains syllabic writings of terms which in other texts are invariably written with Sumerograms (*šiddum* for uš, *nārum* for íd).

54 Texte VI, TMS 49ff.

55 BM 85194 (MKT I 143ff.) and BM 85196 (MKT II 43ff.).

56 Texte IX 5 and 12, and Texte XXI 4 (TMS 63 and 108). The edition transcribes as *šutamḫurum* and translates as *šutākulum*!

57 Cf. above, note 21. The ideographic role of the sign in connection with squaring and "rectangularization" should of course be distinguished from its logographic role inside other semantic fields.

The sign is , a repeated LAGAB. As in the logogram ì-kú-kú, the repetition looks like an intentional graphic repetition of the reciprocity of the Št-stems *šutākulum* and *šutamḫurum* or perhaps a representation of the use of two lines to stretch the square or rectangle. Cf. also note 58 on UL.UL and UR.UR.

58 YBC 4662 and 4663 passim (MCT 69, 71f.). In YBC 4662, the term occurs in the construction *x* a-rá *x* UR.UR.a; however, in several other constructions (appending, i.e. an additive operation; raising) the tablet writes a-rá instead of *ana*, due perhaps to a dictation or writing error; so, I guess that the original intention was *x ana x* .... In YBC 4663, the term when used for squaring gives the factor only once (3°15′ UR.UR.ta), but for once *šutākulum* is used in the same way in that tablet (rev. 20). On the other hand, while the tablet has uš sag UR.UR.ta (ta ~ *ina*, "from"/"by means of"), it writes uš *u* sag *šutākil* (*u* ~ "and"); UR.UR can therefore not be a pure logogram for *šutākulum*, instead the whole phrase is written as an ideographic syncope.

A. Goetze (MCT 148) counts the two tablets among the early Southern ones. Both, however, state results with the word *tammar*, "you see", as do the texts belonging to his group VI and other Northern texts (cf. below, note 84).

As in the case of ì-kú-kú as a logogram for *šutākulum*, the repetitive structure of UR.UR is probably to be read as a (pseudo-) Sumerian rendition of the reciprocity of the Št-form *šutamḫurum* —or, rather, as a way to render in Sumerian grammar a geometrical idea which is rendered in Akkadian by the Št-stem, and rendered badly so, as the verb has only one object.

59 AO 17264 obv., 13f. (MKT I 126, cf. TMB 74).

and for *nakrum* "enemy", probably derived from the association of this concept with *maḫārum* (cf. above).[60] Because of the ideographic but probably not logographic equivalence with *maḫārum* I propose the standard translation "oppose".

UL.UL is found in 7 tablets[61], in all of which it is used for squarings, in a way which could make it a logogram for *šutamḫurum*. But in one of them[62] it is also used in the same role as *šutākulum*, and in another[63] it is also used as a substitute for íb-si$_8$ in a situation where this term could be translated "as a square" or "squared", and where it is kept apart from *šutākulum* and its relatives. So, we have to do with yet another ideogram to which no well-defined logographic value can be ascribed.

Once more, the term appears to point to the idea of confrontation of equal forces. Originally the sign represents a lowered bull's head, corresponding to the reading ru$_5$ (used logographically for *nakāpum*, "to butt"). UL.UL should then be read ru$_5$-ru$_5$, viz. as a logogram for *itkupum* 'to butt each other", "to join battle"[64], and figuratively thus "to confront". Since this latter term is already used, I shall propose a distinct but semantically analogous standard translation, "to make encounter".

IV. 5. Halving

As it is later seen in Medieval elementary arithmetic, halving is a separate operation in Old Babylonian mathematics, or, rather, it occurs as a specific operation in certain specific connections. Chief among these are the bisection of a side or of a sum of opposing sides when areas of triangles or quadrangles are calculated, and the halving of the "coefficient of the first-degree term" in the treatment of second-degree equations. The term used is the verb *ḫepûm*, "zerbrechen", in connections like "break into two" or "half of x break" (where I have used the standard translation "break"). Certain texts use the Sumerogram gaz.

The half resulting from a "breaking" operation is designated *bāmtum* (occasionally abbreviated or Sumerianized BA.A), a term which I shall translate "moiety". It is distinguished from the normal half, *mišlum* (~šu-ri-a), which designates the number $^1/_2 = 30'$ as well as that half of an entity which is obtained through multiplication by 30′.[65]

60 All three values appear to belong originally to UR$_5$, but all are also testified for UR—cf. the terms in question in AHw, and MEA, No 401 (UR$_5$) and No 575 (UR).
It may be worth noticing that the original sign for UR$_5$ still used on stone in the Old Babylonian period was a square standing on a corner: ◈.

61 Str. 363 passim (MKT I 244f.); Str. 368 rev. 5, 8 (MKT I 311); VAT 7532 obv. 19 (MKT I 295); VAT 7535 rev. 17 (MKT I 305); VAT 7620 passim (MKT I 315); YBC 6504 passim (MKT III 22f.).

62 Str. 363 rev. 15f.: . . . 20 *u* 1 UL.UL-*ma* 20 / 40 *u* 5 UL.UL-*ma* 3′20° . . . . Furthermore, in obv. 9 of the same tablet a relative clause refers back to UL.UL by a syllabic *šutākulum*.

63 YBC 6504. In the first two problems of the tablet, íb-si$_8$ is used in the statement, while *šutākulum* is used for squarings in the prescription of the procedure, and íb-si$_8$ turns up when towards the end a square-root is taken. In the third and fourth problems, UL.UL is used both in statement and procedure for squarings, while íb-si$_8$ is still used for the square-root.

64 See CAD *nakāpu*. I am grateful to A. Westenholz for pointing out this meaning of UL.UL to me, whose implications I had overlooked.

65 One place where the distinction between "halving" and "division by 2" (i.e. multiplication by 2^{-1}) is especially obvious is Str. 367 rev. 3f. (MKT I 260). A clear distinction between *bāmtum* and *mišlum* is found in the tablets AO 8862 (below, section XIII.2) and BM 13901 (MKT III 1–5). A single tablet (YBC 6504, MKT III 22f.) uses šu-ri-a where others have *bāmtum*.

According to parallels from other Semitic languages, *bāmtum* was originally a designation for a rib-side or for the slope of a mountain ridge. Probably because such a side or slope can be apprehended as one of two opposing sides or slopes, the term is used in a variety of situations where an entity splits naturally or customarily into two parts, or where e.g. a building is composed of two wings. In mathematical texts, it is used similarly for the semi-sum of opposing sides in a trapezium or the semi-diameter of a circle —all being halves of entities falling naturally or by customary procedure into two "wings".

Below, we shall also see it in an important role in the treatment of second-degree equations (section V.2. on BM 13901 No 1, and passim).

IV. 6. Division

As it is well known, Babylonian mathematics possessed no genuine operation of division. Division was a problem, no procedure. If the divisor b of a problem a/b was regular, i.e. if it could be written in the form $2^\alpha \cdot 3^\beta \cdot 5^\gamma$, in which case its reciprocal would be written as a finite sexagesimal fraction, and if it was not too big, $1/b$ would be found in agreement with the standard table of reciprocals[66], and a/b would be found by "raising" $1/b$ to a. If b was irregular [67], or if it was complicated to be recognized as regular, a mathematical problem text would simply formulate the division as a problem, "what shall I pose to b which gives me a?", and next state the solution–since normal mathematical problems were constructed backwards from known solutions, the ratio would always be expressible and mostly known.

Two concepts are important in connection with the method of reciprocals: That of the reciprocal itself, and that of the process through which it is found. The reciprocal of n is spoken of as igi n gál-bi, at times abridged to igi n gál or simply igi n. The literal meaning of the expression is unclear, but it is

66 The standard table of reciprocals lists the reciprocals of the regular numbers from 1 to 1′21° (=81) (cf. MKT I 9ff.). It can be legitimately discussed whether our term "table of reciprocals" is anachronistic. Indeed, one table, which appears to antedate 1850 B. C. (MKT I 10 No 4), seems to express the idea that not $1/n$ but $60/n$ is tabulated (Scheil 1915: 196). As argued by Steinkeller (1979: 187), another table with phonetically written numbers suggests the same idea (in MKT I 26f.). On the other hand, such conceptualizations of early tables have no necessary implications for the understanding which Old Babylonian calculators had of the tables used in their own times, and two observations combined suggest that they did in fact apprehend their own tables as tabulations of the numbers $1/n$. Firstly, they used the tables for divisions, i.e. for multiplications with these numbers. Secondly, there is textual evidence that they possessed a specific concept for the number $1/n$, as distinct from a general "n'th part" of something (cf. below, note 69).

67 A few tables containing approximate reciprocals of certain irregular numbers exist: YBC 10529 lists reciprocals of regular as well as irregular numbers between 56 and 1′20° (MCT 16). M 10, John F. Lewis Collection, Free Libr. Philadelphia gives reciprocals of 7, 11, 13, 14 and 17 (Sachs 1952, 152). Apparently, however, such approximations are not used in the Old Babylonian mathematical texts, and since the irregular divisors of these texts always divide the dividends, such use would indeed lead to errors which could not go unnoticed.

testified as early as c. 2400 B.C. in the sense of "the n'th".[68] Some Old Babylonian mathematical texts use it both in this general sense as "the n'th of some quantity", and in the special sense of "$1/n$" regarded as a number, but in a way which distinguishes the two.[69] There is therefore no doubt that the Old Babylonian calculators had a specific concept for the number $1/n$, which I shall designate by the standard quasi-translation "igi of n". The general sense I shall render simply by "the n'th part".

To "find" a reciprocal is spoken of by the verb *paṭārum* "(ab)lösen, auslösen" with the logographic sumerogram du_8. In F. Thureau-Dangin's opinion[70], this term should be understood in analogy with the modern metaphor "to solve a problem". However, in two texts the term is also used subtractively[71], in a

[68] VAT 4768 and VAT 4675, published by Förtsch (1916 N^os 65 and 175), transliterated and translated by Bauer (1967: 508–511). The texts belong to the fourth year of Lugalanda, and speak of 1/4 šekel silver and 1/6 šekel silver, by the phrase igi n gál-Similar contemporary evidence (also from Lagaš) is found in Lambert 1953:60, 105, 106, 108, 110 (1/3, 1/4 and 1/6 šekel of silver or lead) and Allotte de la Fuÿe 1915: 132 (1/4 sar of land).–All these tablets antedate the first known occurrences of sexagesimal reciprocals by some 350 years, and they antedate by c. 200 years a school text which suggests that the ideas behind the sexagesimal system were on their way but not yet mature nor formulated around 2200 B.C. (Limet 1973 No 36; cf. commentaries in Powell 1976:426f. and Høyrup 1982:28). We can therefore confidently infer that the general sense of a reciprocal is a secondary derivation. This undermines the only plausible yet grammatically somewhat enigmatic explanation of the term given to date, one offered by Bruins (e.g. 1971:240): Literally, the phrase igi 6 gál-bi 10-àm could mean "in the front of 6 is: 10 is it", i.e. "in front of 6 is found what (in the table of reciprocals)? 10". This explanation would interchange basic and derived meaning, and unless unexpected evidence turns up which moves the tables of reciprocals back into the mid-third millennium, it cannot be upheld. –Truly, Bruins (1983: 105, and earlier) points to two Old Babylonian texts which write the Akkadian term *pāni*, "in front of", in order to designate the reciprocal. (So does also Haddad 104, see al-Rawi–Roaf 1984, section 0.4.3). Certain Old Babylonian scribes hence appear to have held the same hypothesis as Bruins concerning the origin of the expression. But Old Babylonian scribes may as easily have constructed a scholarly pseudo-etymology as they can have guessed correctly a conceptual development which had taken place some 800 years before their own time. In any case, current logographic use of igi for *pānum* may easily have led them astray to an erroneous "folk etymology".

[69] Str. 367 (MKT I 259f.) speaks in obv. 3 of "the third part" of a length in a complete phrase igi 3 gál, while the reciprocals of 4, 1, 3, 2, 3′ 20″ and 1`12° are spoken of (passim) simply as igi n. The same distinction is made in VAT 7532 and VAT 7535 (MKT I 294f. and 303ff.); here, even the n'th part of the number 1 is spoken of in the complete phrase when this number 1 is taken to represent an unknown length, and the part hence understood as a fraction of something, not as a reciprocal (a number). In BM 85210 rev. I 0–12 (MKT I 221f.), the "n'th part of m" is also spoken of by the complete expression and the reciprocals simply by igi n; but furthermore, while the finding of the latter is spoken of by the usual term du_8 ($\sim$*paṭārum*, "to detach", cf. below), the process producing the former is designated by zi ($\sim$*nasāḫum*, "to tear out"). BM 85194 (rev. I 28, rev. III 2f., and passim; MKT I 143ff.) speaks of both "part" and "reciprocal" by means of the abbreviated expression, but distinguishes by means of the differentiation between zi and du_8.

[70] Thureau-Dangin 1936: 56.

[71] In Str. 367 (MKT I 259f.) a triangle of area 21′36° is "detached" from a trapezium of area 36′, leaving a rectangle of area 14′25°. The other subtractive occurrence is Str. 362 obv. 15 (MKT I 240).

way which is only explained by the literal sense "detach". To "find the reciproca of n" is thus to be understood as "to detach the n'th part (from 1)"[72], a phrase that shall be my standard translation.

The division by an irregular number calls for few terminological commentaries. The term "pose" (my standard translation for *šakānum* $\sim$ gar, see below) is no term for multiplication; at times, the multiplication to be performed is implicitly understood in the expression, but more often it is stated explicitly.[73] In the latter cases, the term used belongs invariably to the "raising"-class (*našûm*, íl, nim).

The same was the case when a dividend was multiplied by the reciprocal of a divisor, even when one side of a rectangle is found from the area and the other side.[74] Apart from the (purely arithmetic) distinction between regular and irregular divisors, division is one thing, and it is the inverse of raising. Nothing corresponding to the distinction between four different "multiplications" is found. This could be interpreted as evidence that the Babylonians understood their division as a common, purely arithmetic inversion of all four multiplications, the isomorphism between which they have of course recognized. Still, since such an understanding would rather lead to use of the purely arithmetical term a-rá, it seems to be a better explanation that the real multiplicative operation was "raising", while the other three classes were in reality something else which could not be reversed (as we shall see below, there are good reasons to apprehend "repetition" as real repetition of the concrete entity, and "spanning" as a constructive procedure; neither of these procedures is of course reversible).

IV. 7. Variables, derived variables, and units

Besides the above-mentioned terms for arithmetical operations, a number of basic concepts and appurtenant terms can profitably be presented in advance and briefly discussed. A first group contains the standard names for unknown quantities ("variables"), the way to label new variables, and the units.

By speaking of standard names for unknown quantities I want to emphasize once more that the Babylonians formulated algebraic problems dealing with

[72] Cf. also the subtractive conceptualization of the process "to find the n'th part of m" in BM 85210 and BM 85194 (see note 69).–Further evidence against F. Thureau-Dangin's assumption comes from the way the finding of a square-root is spoken of: You are requested to "make the equilateral come up" *(šūlûm < elûm)*; you "take" it *(laqûm)*; or the question is asked, "what the equilateral" (*minûm* íb-si_8). Had *paṭārum* meant simply "to solve" an arithmetical problem, nothing would have prevented the Babylonians from using it also for the solution of the problem $x \cdot x = A$.

[73] VAT 8389 obv. II 6–9 (below, section VII.1); VAT 8391 rev. I 28–30 (below, section VII.2); VAT 8512 rev. 1–5 (MKT I 341); VAT 8520 obv. 24f., rev. 25f. (MKT I 346f.); Str. 363 passim (MKT I 244f.).

[74] Str. 367 rev. 11 (MKT I 260); VAT 8512 obv. 10–12 (MKT I 341). A possible exception is AO 6770, N° 1, lines 5–7. Still, since no really satisfactory interpretation of this text has been given, it can hardly serve as evidence for anything. Improved transliteration and bibliography of earlier treatments of this text will be found in Brentjes–Müller 1982 (cf. Høyrup 1984 for reasons why even this latest interpretation is problematic).

many types of quantities: Numbers, prices, weights of stones, etc. One set of such unknown quantities, however, belongs with the "basic conceptualization" of Old Babylonian algebra, as unknown abstract numbers represented by letters belong with our own basic conceptualization (cf. chapter I).

These basic variables are of course the length and the width. They form a fixed pair. "Length" translates uš (very rarely written phonetically with the Akkadian term *šiddum*, "Seite, Rand; Vorhang"). "Width" translates sag, literally "head, front" (the rare corresponding Akkadian term is *pūtum*).[75] Both terms appear in surveying texts from Early Dynastic Lagaš[76]; surveying is thus the distant point of origin of the Old Babylonian second-degree algebra which should not necessarily be confused with its Old Babylonian conceptualization.

Problems in only one variable are basically formulated as concerned with a square identified with its side: *mitḫartum*, LAGAB, or NIGIN (see above, section III.4, "squaring and square-root"). In two Susa texts, the side of the square is occasionally spoken of explicitly as uš, "length", of the "square figure".[77]

In problems in one as well as two variables, the "second-degree-term" is spoken of by the same expression, a-šà, "field". Like "length" and "width", it is almost invariably written by the sumerogram, but in a number of places it occurs with a phonetic complement indicating a purely logographic use for the Akkadian *eqlum*.[78], [79] I shall use the standard translation "surface" as I want

[75] Strictly speaking, the Akkadian terms are not just rare. Excepting the Tell Ḫarmal compendium (which has uš ~ *šiddum*, but on which see note 53) they are never used as names for the standard variables but only in a couple of texts dealing with real rectangles: Db$_2$–146, obv. 3 (in Baqir 1962: Pl. 3; *šiddum* alone) and IM 53965, passim (in Baqir 1951; both terms). On the use of *pātum* (plural of *pūtum*) to designate the sides of a real square in BM 13901, No 23, cf. below, section V.4. Three final occurrences deal with carrying distances for bricks and the width of a canal.

[76] See the texts from c. 2400 B.C. published and discussed by Allotte de la Fuÿe (1915). A difference between the Early Dynastic surveying texts and the Old Babylonian standard algebra problems should be noted: While the latter tell us that they deal with a rectangle simply by speaking of uš and sag without any epithet, implying thereby that there is only one length and one width, the former will normally present all four sides of a quadrangle, and if a pair of opposing sides are equal they will with one exception which seems most hastily written tell explicitly that this is uš si$_8$, "lengths being equal", or sag si$_8$, "widths being equal".

[77] Even though the length is spoken of explicitly, the same lines of the text will identify the "confrontation" (LAGAB) itself with a number, viz. with the same number as the "length". Here as everywhere, square figure and side are conceptually conflated. So TMS V, obv. II.1: "The CONFRONTATION *and* 1/11 of *my* length accumulated: 1", i.e. confrontation = length = 55′.

[78] On the other hand, the terms uš and sag are on the same and other sorts of evidence not real logograms for *šiddum* and *pūtum* (cf. above).

[79] Like uš and sag, a-šà is used already in Early Dynastic texts (cf. note 72). It seems plausible that this rooting in an old tradition should be linked causally with the all-dominating Sumerographic writing (in fact, full phonetic writing of *eqlum* is as absent as phonetic Akkadian writing of uš and sag). In contrast, the unknown "confrontation" in problems of one unknown is not written by the traditional Sumerogram si$_8$ (cf. note 45). This appears to indicate that theoretical algebraic problems among which the problems of one unknown are important did not arise until the Old Babylonian age,

to avoid the connotations associated with the word "area": A number which describes or measures a surface. Such distinction between entity and measuring number is apparently not true to Babylonian thought.

A number of texts use terms like "length", "width" or "surface" for a succession of different numbers (in cases where we would use successively x and $\bar{x}$, etc.). In such cases the two different "lengths" can be distinguished by an epithet appended to one of them: lul corresponding to Akkadian *sarrum*, which is used in TMS XI and XXIV; standard translation "false") or *kīnum* (~gi-na; standard translation "true"). The use of these terms is best elucidated in connection with their occurrence in specific texts.

Another term with a related function is kúr, a Sumerogram used logographically for *nakārum*, "anders, fremd, feindlich sein, werden", and for its various derivatives. It turns up in certain series texts when a "second" or "modified" width occurs besides the width first considered. I shall propose the standard translation "alternate".

In contrast to Modern algebra, the seemingly pure numbers reveal themselves in certain texts as numbers counting a multiple of the basic unit of length, the nindan[80] (1 nindan equals c. 6 m). In problems concerned with volumes, however, the vertical dimension is measured in "cubits" (*ammatum* ~ kuš = $^1/_{12}$ nindan), even when the problem is nothing but "disguised algebra". Areas are measured correspondingly in the unit sar = nindan^2, volumes in (volume-) sar = nindan^2 · kuš[81], i.e., a surface of 1 sar covered to the height of 1 kuš.

IV. 8. Recording

A large number of terms are used when given quantities and intermediate and final results are announced and taken note of. Some of them are mutually distinct, some are used inside the mathematical texts as "practical synonyms" (although they are not synonymous in their general use).

Most important is *šakānum*, "hinstellen, (ein)setzen, anlegen; versehen mit", and its Sumerogram gar. It may well have a precise technical meaning in the mathematical texts, but since this sense can only be approached by indirect means, I shall use a semantically rather neutral standard translation, "to pose".

The term is often used after the statement of a problem, when the given numbers are "posed" before calculations begin–they appear to be taken note of in some manner as a preparation for operations. Similarly, intermediate results are occasionally "posed" (but then mostly "posed to" or "posed by" a length etc.,

or at least that they arose among Akkadian speakers—in which connection it may be of interest that a specific Akkadian record-keeping system, distinct from the contemporary Sumerian system, was in use during the Sargonic era (see Foster 1982: 22–25). A similar conclusion could be drawn from the greater part of the basic algebraic vocabulary, which is written alternatingly in phonetic and ideographic writing, but where the latter writing is reconstructed and not traditional Sumerian.

[80] Written GAR in MKT and NINDA in TMB. Cf. Powell 1972: 198f. on the transliteration nindan.

[81] More complete information on the Old Babylonian metrological system will be found in TMB (pp. xiii–xvii) and MCT (pp. 4–6).

cf. below). In one case, even final results are recorded by "posing".[82] Finally, the term is invariably used in divisions by an irregular divisor, cf. above, section III.6.

The recording of intermediate results can also be spoken of by the verb *lapātum*, "eingreifen in, anfassen, schreiben" (rarely, it can also be used for the recording of a given number).[83] I shall use the standard translation "to inscribe".

The verb *nadûm*, "werfen, hin- niederlegen", is used in two apparently different functions, one of which might look as a "practical synonym" for *šakānum* and *lapātum*. In some texts, when the "equilateral" (i.e. square-root), of a number has been found, it is "laid down" in two copies, to one of which is added, and from the other of which is subtracted.[84] Two texts use "posing" in the same function, and four employ *lapātum* in a related way.[85] On the other hand, however, *nadûm* is never used in the other functions of these terms.

The other use of *nadûm* is in the tablet BM 15285[86], where the drawing of indubitably geometrical squares, circles and triangles is referred to by the term.

Even outside the domain of mathematical texts, similar uses of the term are known: "Bauten usw. anlegen"; "(Fang)netz auslegen"; "(auf Tafel usw.) eintragen, einzeichnen"; "Grundriß aufzeichnen".[87] I shall use the standard translation "to lay down", which shall therefore be read as "to lay down (in writing or drawing)". Since the former use is restricted to the laying down of entities which in the geometrical interpretation of the texts are the sides of squares, it is my guess that the real meaning in all mathematical texts is simply "to draw".

A specific phrase for recording an (invariably intermediate) result is *rēška likīl* "may your head retain (it)" (from *rēšum*, "Kopf, Haupt; Anfang, ...", and *kullum* "(fest)halten"). Apparently, the term is reserved for the storing of intermediate results of linear transformations (cf. below, section VII.2.).

The appearance of a result can be announced in various ways. It can be said that a number "comes up for you" (standard translation of *illiakkum*, from *elûm* "auf-, emporsteigen", Stative "hoch sein"), or that a calculation "gives" a certain result (my standard translation of *nadānum* "geben"; and of the Sumerogram sum). Finally, the result can be announced by the term *tammar* "you see" (from *amārum* "sehen"). The choice appears to depend exclusively on the geo-

[82] YBC 6504, passim (MKT III 22f.). In the same text, intermediate results too are "posed".

[83] IM 52301 obv. 19f. (below, section X.1); the text is rather late and contains several other deviations from normal usage); IM 54478 obv. 7 (Baqir 1951: 30). In the newly discovered text from Tell Haddad (Haddad 104 IV 9, 17, 29; in al-Rawi–Roaf 1984) the form *lupput* (D-stem, stative) is used of numbers which "stand written down" in a table of constant factors.

[84] VAT 8520 obv. 21, rev. 20 (MKT I 346f.); YBC 6967 obv. 11. Cf. below sections V. 1 and VIII. 4. A slightly different phrasing is found in IM 52301 rev. 5 and 10 (cf. note 79) and in Db_2 –146, 4 and 13 (Baqir 1962: Pl. 3), and another possibly in TMS XVII 12.

[85] "Posing" stands precisely as *nadûm* in TMS XIII, 10 (cf. correction to the line in von Soden 1964:49) and in IM 53965 rev. 7 (Baqir 1951:39). In AO 8862 II 21f. (MKT I 110), BM 13901 obv. II 8 (MKT III 2), YBC 4662 obv. 21 and 33 (MCT 71), and in YBC 4663 rev. 23 (MCT 69), finally, the "equilateral" is "inscribed until twice".

[86] Most recent edition with addition of a large fragment in Saggs 1960.

[87] AHw, article *nadû(m)* III, §§ 20, 22, 24.

graphical and chronological origin of the text (and in certain texts perhaps on personal taste).[88] The mathematical functions of all three coincide.

Very often, a result appears simply as a number, announced by no special word or at most by the enclitic particle *-ma* appended to the foregoing phrase. A single text uses the Sumerogram for "posing", gar (cf. above, note 82).

IV. 9. Structuration

The terms discussed till here were all concerned with the "arithmetical" level of the texts, that of single calculations. Another group of terms belongs to the meta-level which makes the texts "algebraic", and which structures the texts.

All those texts which describe a problem together with its solution start by stating the problem, after which the procedure is described. The former is written in the first person (viz. the teacher), past tense (only the excess of length over width will invariably be stated in the present tense). The procedure is formulated in the second person (the student), present tense, or the imperative, by a person (the instructor) who refers to the teacher in the third person. The statement has no special name, but the procedure is designated *epēšum* with Sumerographic equivalent kì. The term is the infinitive of a verb („machen, tun; bauen") used as a noun; when the description is finished, the derived term *nēpešum* is used. For *epēšum* I shall use the standard translation "the making", for *nēpešum* "the having-been-made".

Inside the description of the procedure, the statement of the problem may be quoted in justification of certain steps being made. This is done by the phrase "he has said", using the verb *qabûm*, sagen, „befehlen", which functions simply as a quotation mark.

Three terms are traditionally interpreted as indications that we pass from

[88] sum and *nadānum* are found in the texts to which A. Goetze ascribes for linguistic reasons an early, southern origin (groups I–IV, see MCT 146–151). *tammar* is found in his group VI ("northern modernizations of southern (Larsa) originals"), in the Susa texts of TMS and in a number of the late (and northern) Tell Ḥarmal texts (in Baqir 1950a and 1951); the early Tell Ḥarmal text IM 55357 (Baqir 1950:41–43) uses igi-dù, a logogram for *tammar*, mistaken by homophony for igi-du_8, which is used in the same function in YBC 4669 (rev. I 5–7; MKT III 27) and YBC 4673 (rev. III passim; MKT III 31); these too are probably northern, cf. MKT I 387f. and 123f. *illiakkum* and related derivations from *elûm* are found in Goetze's group V ("northern characteristics", maybe somewhat older than the group VI texts); in the remaining late Tell Ḥarmal texts (Baqir 1951); and finally in the early northern texts Db_2–146 (Baqir 1962: Pl. 3) and Haddad 104 (al-Rawi – Roaf 1984). – Only very few exceptions to these clear-cut rules are found. The group I text YBC 7997 (MCT 98) aligns *nadānum* and *elûm*, the former being used for final results alone; another group I text (YBC 4675, with the parallel fragment YBC 9852–MCT 44f.) uses *elûm* exclusively. *tammqr* is used alongside with *nadānum* in YBC 4662, which A. Goetze locates in his group II (Larsa?), and it is used alone in MLC 1950 (MCT 48), which shares a specific Sumerian standard phrase with a number of texts belonging to group III but is otherwise unlocated. Finally, *tammar* and *elûm* are found together in one late Tell Ḥarmal text (IM 54559; Baqir 1951:41), while igi alone is found in VAT 672 (MKT I 267), a fragment with other stylistic peculiarities and containing too little Akkadian to allow for linguistic analysis.

one section of the statement or the procedure to the next: *saḫārum* ("sich wenden, herumgehen" etc.), *târum* ("sich umwenden, umkehren, zurückkehren; (wieder) werden zu"), and nigín(-na) (the sign LAGAB). which on other text-types is testified as a logogram for both. However, as first pointed out to me by A. Westenholz in connection with the use of *saḫārum* in AO 8862 (see section VIII.2), the denotation of this term and mostly also of *târum* appear to be much more precise and concrete, viz. real movement around a field. This also fits some of the occurrences of nigín but not all of them; it seems that the same technicalization which has led to logographic writing and, apparently, to conflation of the two terms as synonyms covered by the same logogram, has also reduced it to a textual delimiter. In order to make these distinctions visible I shall use the standard translations "go around" for *saḫārum* and "turn back" for both *târum* and nigín.

The hypothetical-deductive structure of the complex problem+procedure may be expressed by terms like *šumma* ("wenn, falls", standard translation "if"–also the recurrent term of the hypothetico-deductive omen texts), *inūma* ("als, wenn usw."; standard translation "as") and *aššum* ("wegen, weil usw."; standard translation "since"). Most often, it is left implicit–the statement appears as a fact, and after a phrase "You, by your making" comes an equally descriptive (occasionally jussive) procedure-part.

The equality necessary to establish an equation is normally implied the particle *-ma* followed by a numerical value (the "right-hand side" of the equation) (cf. above, chapter II). As stated there, I shall render *-ma* by the sign ":". If two unknown quantities are equated, the term *kīma* ("wie; als, wenn, daß", standard translation "as much as") can be found.

A term for equality which may function as sort of bracket is *mala* ("entsprechend (wie), gemäß;" standard translation "so much as"), used in the expression "so much as x over y goes beyond", meaning $(x-y)$.

The numerical value of a quantity can be asked for in two ways, either by the question "x *mīnum*" (*mīnum*, "was"; standard translation "what"; Sumerographic equivalent en-nam) or by a question like "*kī maṣi* x" (*kī*, „wie, als, daß"; *maṣûm*, "entsprechen, genügen, ausreichen"; standard translation of the combined expression "corresponding to what"). In a few texts, the student is asked to "make the equilateral (square-root) of x come up" (x *basā-šu šūli*).

IV. 10. The "conformal translation"

Obviously, the shades and distinctions just described in IV. 1 to IV. 9 cannot be rendered in a translation, in particular not in a translation into a non-Semitic language. One cannot achieve at the same time a one-to-one correspondence for single terms and an acceptable English sentence, not to speak of the rendition of grammatical categories. It is thus for good reasons that O. Neugebauer restricted the role of the translation to that of a general guide, "selbstverständlich genau genug, um den Inhalt korrekt erfassen zu können, nicht aber, um die Feinheiten der Terminologie und Grammatik daran ablesen zu können".[89]

[89] MKT I, viii. MKT III 5 continues "Wer terminologiegeschichtliche Studien an Hand einer Übersetzung machen will, dem ist doch nicht zu helfen".

Therefore, an investigation of Babylonian mathematics which tries to go beyond mathematical contents and penetrate patterns of thought and conceptualizations must necessarily rely on texts in the original language. On the other hand, the presentation of the results at least to the non-assyriologist must by the same necessity approach the question through a modern language.

Since the results of my investigation can only be documented and partly only explained with reference to original texts, translations are necessary. Since, on the other hand, the translations cannot be allowed to loose those shades and distinctions which cannot be translated into idiomatic English, I have chosen a compromise somewhere between a code and a real translation: All words except a few key terms are rendered by English words; a given expression is in principle always rendered by the same English expression, and different expressions are rendered differently with the only exception that well-established logographic equivalence is rendered by coinciding translation but distinct typography, while possibly mere ideographic equivalence is rendered by translational differentiation. Terms of different word class derived from the same root are rendered (when the result is not too awkward) by derivations from the same root.[90] These translations are the "standard translations" presented above. Furthermore, syntactical structure and grammatical forms are rendered as far as possible by corresponding structure and grammatical forms; the simple style of the mathematical texts makes this feasible. Expressed in mathematicians' *argot*, this sort of pseudo-translation could be called a "conformal translation".

Each line of the translation is followed by a transliteration of the original text. Here, as in current usage, phonetic Akkadian is written in italics. Sumerian words and Sumerograms (i.e., Sumerian words used logographically or ideographically for Akkadian speech) are given in spaced writing; and signs which can neither be interpreted one way or the other either because they should not be, or because our knowledge is insufficient are written in small capitals. In order to follow the principle of conformity as far as possible, and in order to facilitate the comparison of translation and transliteration, the same typographical distinctions are used in the translation. So, *kamārum* is translated "to *accumulate*"; gar-gar will be found as "to accumulate" (or another adequate form–often Sumerograms etc. are found with no phonetic or grammatical complements indicating which grammatical form to choose); and UL.GAR is rendered "to ACCUMULATE". Ideograms written with an Akkadian phonetic complement are translated in mixed writing. So, a-šà*[lam]* is translated as "sur*face*". The result violates all ideals of typographic beauty, but it should make it relatively easy for the reader who wants to do so to acquire quickly a rudimentary feeling of the original formulation.

According to analogous considerations, each number is rendered in the translation the way it stands in the original text: Standard sexagesimal numbers are written in the extended degree-minute-second-notation described in note 4. In the transliteration, the same numbers are given more faithfully, with no indication of absolute place. Number words, including words for ordinal numbers and fractions, are rendered by words. Special signs for fractions are written as

[90] So, *epēšum* and the logogram kì when used as verbs are rendered "to make", the infinitives used as nouns by "the making", and *nēpešum* by "the having-been-made".

modern fractional symbols, $^1/_2$, $^1/_3$ etc. Ordinals and fractions written on the tablet as a number followed by a phonetic or grammatical complement are written 1st, 2nd, etc.

Of course, considerations of intelligibility put some constraints on the principle of conformity. Prepositions cannot always be rendered in the same way, nor can a number of particles which structure the Akkadian sentences (relative pronouns etc.). Certain details of the syntactical structure (e.g. the postpositive adjective) have to be given up. Furthermore, definite and indefinite articles and other English grammatical elements have to be inserted into the translation. Such insertions stand as normal writing, without spacing, emphasis and capitals.[91] In the case of ideograms without complements even markings of grammatical person etc. are written that way. Other, genuine explanatory insertions are given as normal writing in parenthesis.

In the transliterations, all restitutions of damaged passages are of course indicated by square brackets. In order not to make the typographical appearance of the translations too disorganized, I have omitted there all indications of such restitutions, when they are taken over from the original publications of the texts, and when I find them firmly established. Since the restitutions of MKT, TMB and MCT were made with great care, mainly from parallel passages of the same tablets, this holds for most restitutions. Restitutions for which I am responsible myself and restitutions which I consider more or less uncertain are indicated clearly even in the translations.

The English terms used as standard translations of Akkadian terms are normally chosen in a way which respects the use of the latter in non-mathematical texts, and which at the same time shows the possible metaphorical use of the term in a mathematical context. A possible alternative would have been a translation by modern technical terms (e.g. "plus" for *kamārum* "added to" for *waṣābum* "multiply$_1$", "multiply$_2$", . . ., "multiply$_\nu$" for the variety of multiplicative operations and terms). The point of my choice is not that the Akkadian terms were necessarily used as metaphors and not technically. It is that the technical function of a Babylonian term must be learnt from its own context, not by imposition from the outside of inadequate, modernizing categorizations. Indeed, one need not work for very long with a term like "to append" before one forgets most of the concrete connotations and apprehends its single occurrences technically.

The basic vocabulary for arithmetical operations, for the announcement and recording of given numbers and results and for the structuration of the texts was presented above together with the standard translations of the single terms. For the sake of clearness, it is listed again in short form in Table 1, where the ordering corresponds to the above discussion. Table 2 lists all terms for which a standard translation is used in the translations of sections V–X, ordered alphabetically according to the standard translations. Table 3 contains the same material but ordered alphabetically according to the transliterated original language.

[91] So, in a genitive construction like íb-si$_8$ 15′, the preposition "of" is given in normal writing, "the equilateral of 15′". *mišil* uš will be translated "*half of* the length", because the construct state *mišil* indicates a genitive construction, although no genitive marker is joined to uš.

Table 1. Basic vocabulary

Akkadian	Sumerian etc.	Standard translation	use
1. additive operations			
waṣābum	daḫ	to append	"identity-conserving addition"
kamārum	gar-gar /UL.GAR	to accumulate	"identity-cancelling addition"
kimrātum		things accumulated	sum by *kamārum* etc.
nakmartum		accumulated	”
kumurrûm	gar-gar /UL.GAR	accumulatiou	”
2. subtractive operations			
eli . . . watārum	ugu . . . dirig/SI	over . . . go beyond	"subtraction" by comparison
nasāḫum	zi	to tear out	"subtraction" by removal
ḫarāṣum		to cut off	”
3. multiplicative operations			
	a-rá	steps of	number times number
eṣēpum	tab	to repeat	multiplication by positive integer (concrete repetition)
našûm	íl	to raise	calculation by multiplication
	nim	to lift	”
šutākulum	ì-kú(-kú)	to make span	"multiplication" of a "length" by a "width" ("rectangularization")
takīltum		*takīltum*	cf. below, sections V. 1–2
4. squaring and square-root			
	íb-si$_8$	equilateral/ to make equilateral	square-root; geometrical square identified with the length of the side
(maḫārum)		to confront	equality of value, shares (etc.)
šutamḫurum		to make confront itself	formation of a square
mitḫartum	LAGAB(?)	confrontation	square identified with the side
meḫrum	gaba(-ri)	counterpart	"second side of a square"
	NIGIN	to make surround/surrounding	like *šutamḫurum*, *mitḫartum* and (rarely) *sutākulum*
	UR.UR	to oppose	like *šutamḫurum* (and *šutākulum*)
	UL.UL	to make encounter	like *šutamḫurum*, íb-si$_8$ (and *šutākulum*)

Akkadian	Sumerian etc.	standard translation	use
5. halving			
ḫepûm	gaz	to break	bisection
bāmtum	ba/BA.A	moiety	"natural half"; result of bisection
6. division			
(igûm)	igi *n* (gál (-bi))	igi of *n* /*n*'th part	The fraction 1/*n* considered as a number / 1/*n* of something
paṭārum	du$_8$	to detach	To find the reciprocal (to take out 1/*n* from 1)
7. variables, derived variables, units			
(šiddum)	uš	length	one of the two basic "variables"
(pūtum)	sag(-ki)	width	the other basic "variable"
mitḫartum	LAGAB	confrontation	the "variable" in second-degree problems of 1 unknown
	NIGIN	surrounding	"
(eqlum)	a-šà	surface	product, square, and any quantity which in a geometric interpretation *is* a surface
	lul	false	(optional) epithet to a length, width etc. different from the one first considered
kīnum	gi-na	true	(optional) epithet which designates a return to the original use of a term[a]
(nukkurum)	kúr	alternate	a second "variable" within a category already in use
	nindan	nindan	unit of horizontal length, c. 6 m
ammatum	kùš	cubit	1/12 nindan, unit of height and depth, c. 50 cm
	sar	sar	nindan^2 / nindan^2·kuš
8. recording etc.			
šakānum	gar	to pose	presumably material notation and/or drawing, cf. above
lapātum		to inscribe	"
nadûm		to lay down	"
rēška likīl		may your head retain	memorization of intermediate results in linear transformations
illi(-akkum)		comes up (for you)	announcements of a result
nadānum	sum	to give	"
tammar	igi-du$_8$/dù	you see	"

Akkadian	Sumerian etc.	standard translation	use
9. structuration			
epēšum	kì	to make/making	designates the procedure to be used to solve a problem
nēpešum		having-been-made	designates the procedure when performed
qabûm		to say	quotation mark
saḫarum		to go around	apparently the pacing off of a field, by which its dimensions are found
târum	nigin(-na)	to go back	designates the passage to another part of the procedure – concretly or abstractly
šumma		if	marks a deductive structure
inūma		as	"
aššum		since	"
kīma		as much as	equality
-ma		:/that	after verbs: consecution, consequence (result, equality); after nouns: emphasis
mala	a-na	so much as	a rhetorical "bracket"; equality
mīnum	en-nam	what	asks for a value
kī maṣi		corresponding to what	"

[a] In one geometrical text (YBC 8633, in MCT 53), the term "true length" designates that side of a triangle which is closest to being perpendicular to the "width".

Table 2. The standard translations ordered alphabetically

The table is intended to be comprehensive with regard to the texts translated below. Only pronouns and pronominal suffixes are left out intentionally.

The table includes a number of terms which were not represented in the below translations, but which would be useful for other texts belonging to the genre. For this open-ended enterprise, no completeness was of course aimed at.

It should be emphasized once more that this is a table of standard translations, i.e. a key to the translated texts. It is not meant to be a dictionary, and no listing of meanings.

:	*-ma* (after a verb)	bring, to	*wabālum*
accumulate, to	*kamārum*/gar-gar/UL.GAR	build, to	*banûm*
accumulated, the	*nakmartum*	bur	*burûm*/bùr$^{(\text{gán})}$
accumulation	*kumurrum*/gar-gar/UL.GAR	by	*ina*/-ta
add	(Seleucid: *ṭepû*/tab)	change	*takkirtum*
Akkadian	*akkadûm*	collect (taxes, rent) to	*makāsum*
alternate	kúr	come up, to	*elûm*
and	*u*	confront, to	*maḫārum*
append, to	*waṣābum*/daḫ	confrontation	*mitḫartum*/LAGAB/íb-si$_8$ in series texts
appended, the	*wuṣubbûm*	contribution	*manātum*
as	*inūma*	corresponding to what	*kī maṣi*
as much as	*kīma*/gim (nam)	counterpart	*meḫrum*/gaba(-ri)
ask, to	*šâlum*	cubic equilateral	ba-si$_8$/-si
bandûm	*bandûm*	cubit	*ammatum*/kuš
break, to	*ḫepûm*/gaz	cut away, to	*kašātum*
break off, to	*ḫaṣābum*		

cut down, to	*nakāsum*/kud
cut off, to	*ḫarāṣum*
detach, to	*paṭārum*/du$_8$
diminish, to	*maṭûm*
each	ta-àm
equilateral	íb-si$_8$/-si//*basûm*//ba-si$_8$/-si
false	*sarrum*/lul
first . . . second	*ištēn . . . šanûm*
(1st . . . 2nd)	1(kam) . . . 2(kam)
follow, to	*redûm* (as "to make follow", *ruddûm*)
four	*erbûm*
fourth (part)	*rabītum*
from	*ina*/-ta
front	*pūtum*
gin	*šiqlum*/gín
give, to	*nadānum*/sum
go, to	*alākum*/rá
go around, to	*saḫārum*
go away, to	*tebûm*
go beyond, to	*watārum*/dirig/SI
grain	*še'um*/še
great, (to be (come))	*rabûm*/gal
gur	gur
half	*mišlum*/šu-ri-a
hand	*qātum*
having-been-made	*nēpešum*
head	*rēsum*/sag
head retain, may your	*rēska likil*
here	*annikī'am*
if	*šumma*
igûm	*igûm*/igi
igibûm	*igibûm*/igi-bi
inscribe, to	*lapātum*
inscription	*nalpattum*
inside	*libbum*
integrity	*šulmum*
itself	*ramānišu*
know, to	*edûm*/zu
lay down, to	*nadûm*
leave, to	*ezēbum*/tag$_4$
left-over	*šittatum*
length	(*šiddum*)/uš
lift, to	nim
lower	*šaplûm*/ki(-ta)
make, to/making	*epēšum*/kì
make confront itself, to	*šutamḫurum*
make cubic, equilateral	(-e) ba-si$_8$/-si
make encounter	UL.UL
make equilateral	(-e) íb-si$_8$/-si
make follow (additively), to	*ruddûm* (D-stem of *redûm*)
make span, to	*šutākulum*/ì-kú(-kú)
make surround, to	NIGIN
meadow	*tawirtum*/garim
mina	*manûm*
moiety	*bāmtum*/BA.A
name	*šūmum* (Seleucid MU)
nindan	nindan (=GAR)
no(t) (negating a word or part of proposition)	*la*/nu
not (negating a proposition)	*ul(a)*/nu
now	*inanna*
one	*ištēnum*
one . . . the other	*ištēn . . . ištēn*
oppose, to	UR.UR
out from	*ištu*
over	*eli*/u-gù
over-going	*elēnu*
part, *n*'th	igi *n* (gál(-bi))
pose, to	*šakānum*/gar
raise, to	*našûm*/íl
reed	*qanûm*/gi
remain, to	(Seleucid: *riāḫu*)
remainder	*šapiltum*/íb-tag$_4$
repeat	*eṣēpum*/tab
retain, to	*kullum*
sar	sar
say, to	*qabûm*/dug$_4$
saying	dug$_4$/TUK
second/2nd	*šanûm*/2(kam)
see, to	*amārum*, cf. "you see"
seventh	*sebītum*
sila	*qa*/sìla
since	*aššum*
so	*kīam*
so much as	*mala*/a-na
span	see "make span"
steps of	a-rá
surface	*eqlum*/a-šà
surrounding	NIGIN
take	*laqûm*
takiltum	*takīltum*
tear out, to	*nasāḫum*/zi
that	*-ma* (after a noun)
that of	*ša* ("emphatic genitive")
things accumulated	*kimrātum*
third (part)	*šalšum*
three	*šalašum*
threescore	*šuššum*
to (prep.)	*ana*/-ra
together with	*itti*
trapezium	sag-ki-gu$_4$
true	*kīnum*/gi-na
turn back, to	*târum*/nigín(-na)
twice	*šinišu*
two	*šina*

two-third	*šinipâtum*
until	*adi*
upper	*elûm* /an(-ta/-na)
various (things)	ḫi-a/ḫa
wāṣitum	*wāṣītum*
wāṣûm	*wāṣûm*
what	*mīnum*/en-nam
which	*ša*
width	(*pūtum*)/sag(-ki)
you see	*tammar*/igi-dù/-dug$_8$

Table 3. Sumerian and Akkadian terms with equivalences and standard translations

Cf. introductory remarks to Table 2. Only logographic equivalences testified in mathematical texts are listed.

In the translations of the texts, each term is written in the same typography as the (transliteration of the) term it translates.

adi	until
A-ENGUR (~*tawirtum*?)	meadow (?)
akkadûm	Akkadian
alākum (~rá)	to go
amārum	to see
ammatum (~kùš)	cubit
an(-ta/-na) (~*elûm*)	upper
ana (~-ra)	to
a-na (~*mala*)	so much as
annikī'am	here
a-rá	steps of
a-šà (~*eqlum*)	surface
aššum	since
BA(.A) (~*bāmtum*)	moiety
ba-si$_8$/si (~*basûm*)	(cubic) equilateral
bāmtum (~BA.A)	moiety
bandûm	bandûm
banûm	to build
basûm (~ba-si)	equilateral
bùr^{gán} (~*būrum*)	bur
būrum (~bùr^{gán})	bur
daḫ (~*waṣābum*)	to append
(ugu . . .) dirig (~*eli . . . watārum*)	(over . . .) go beyond
du$_8$ (~*paṭārum*)	to detach
dug$_4$ (~*qabûm*, TUK)	to say/saying
edûm (~zu)	to know
elēnu	over-going
eli (~u-gù)	over
elûm (~an)	upper
elûm	to come up (as a result)
en-nam (~*mīnum*)	what
eperum (saḫar)	earth
epēšum (~kì)	to make/making
eqlum (~a-šà)	surface
erbûm	four
eṣēpum (~tab)	to "repeat"
ezēbum (~tag$_4$)	to leave
gaba(ri) (~*meḫrum*)	counterpart
gal (~*rabûm*)	great
GAM (~*šuplum*)	depth
GAM (a-rá)	(Seleucid) steps of
gar (~*šakānum*)	to pose
gar-gar (~*kamārum*)	to accumulate/accumulation
garim (~*tawirtum*)	meadow
gaz (~*ḫepûm*)	to break
gi (~*qanûm*)	reed
gim (nam) (~*kīma*)	as much as
gín (~*šiqlum*)	gin
gi-na (~*kīnum*)	true
giš	giš (=1' nindan)
gur	gur
ḫa = ḫi-a, q. v.	
ḫarāṣum	to cut off
ḫaṣābum	to break off
ḫepûm (~gaz)	to break
ḫi-a	various (things)
íb-si$_8$	(make) equilateral (in statements of series texts: ~*mitḫartum*)
íb-tag$_4$ (~*šapiltum*)	remainder
igi (~*igûm*)	igûm
igi *n* (gal-bi)	igi of *n*/*n*'th part
igi-bi (~*igibûm*)	igibûm
igibûm (~igi-bi)	igibûm
igi-dù/-dug$_8$ (~*tammar*)	you see
igûm (~igi)	igûm
íl (~*našûm*)	to raise
ina (~ -ta)	from/by
inanna	now
inūma	as
ištēnum	one
ištēn . . . ištēn	one . . . the other

ištēn . . . šanûm	the first . . . the second
ištu	out from
itti	together with
-kam	(ordinal suffix)
kamārum (~gar-gar, ~UL.GAR)	to accumulate
kašātum	to cut away
ki (~*itti*)	together with
KI (~*qaqqarum*)	ground
ki(ta) (~*šaplûm*)	lower
kì (~*epēšum*)	to make/making
kī maṣi	corresponding to what
kīam	so
kīma (~gim(nam))	as much as
kimrātum	things accumulated
kīnum (~gi-na)	true
kú-kú (kú in series texts)	to make span
kud (~*nakāsum*)	to cut down
kullum	retain
kumurrûm (~gar-gar; ~UL.GAR)	accumulation
kúr (~*nukkurum*?)	alternate
kùš (~*ammatum*)	cubit
la (~nu)	not
LAGAB (~*mitḫartum*)	confrontation
lapātum	to inscribe
laqûm	to take
libbum	inside
lul	false
-ma (after a verb)	:
-ma (after a noun)	that
maḫārum	to confront
makāsum	to collect (taxes, rent)
mala (~a-na)	so much as
manātum	contribution
manûm	mina
maṭûm (~lal)	to diminish
meḫrum (~gaba(-ri))	counterpart
mīnum (~en-nam)	what
mišlum (~šu-ri-a)	half
mitḫartum (~LAGAB: in series texts ~ib-si$_8$)	confrontation
nadānum (~sum)	to give
nadûm	to lay down
nakāsum (~kud)	to cut down
nakmartum	accumulated
nalpattum	inscription
nasāḫum (~zi)	to tear out
našûm (~il)	to raise
nēpešum	having-been-made
NIGIN	to make surround/surrounding
nigin, nigín	see šu-nigin, šu-nigín
nigín(-na) (~*târum*)	to turn back
nim (~*ullûm*?)	to lift
nindan	nindan
nu (~*la*, ~*ul(a)*)	not
paṭārum (~du$_8$)	to detach
pūtum (cf. sag)	front
qa (~sìla)	sila
qabûm (~dug$_4$)	to say
qanûm (~gi)	reed
qaqqarum (~KI)	ground
qātum	hand
-ra (~*ana*)	to
rá (~*alākum*)	to go
rabītum	fourth (part)
rabûm (~gal)	great
ramānišu	itself
redûm, see *ruddûm*	
rēška likīl	may your head retain
rēšum (~sag in certain contexts)	head
riāḫu (Seleucid)	remain
ruddûm	to make follow (additively)
sag (~*rēšum*)	head
sag (-ki)	width
sag-ki-gu$_4$ (~*absammikum*?)	trapezium
saḫārum	to go around
sar	sar
sarrum (~lul)	false
sebītum	seventh (part)
SI (~dirig, ~*watārum*)	go beyond
sìla (~*qa*)	sila
sum (~*nadānum*)	to give
ṣeḫērum (~tur)	to be (come) small(er)
ṣeḫrum (~tur)	small
ša	which/that of
šakānum (~gar)	to pose
šalašum	three
šalšum	third (part)
šâlum	to ask
šanûm	second
šapiltum (~íb-tag$_4$)	remainder
šaplûm (~ki-ta)	lower
še (~*še'um*)	grain
še'um (~še)	grain
šiddum (cf. uš)	length
šina	two
šinedâtum	two-third

šinišu	twice
šiqlum (~gín)	gin
šittatum	left-over
šū	that
šulmum	integrity
šumma	if
šūmum (Seleucid ~MU)	name
šu-nigin, šu-nigín	total
šuplum (~GAM)	depth
šu-ri-a (~*mišlum*)	half
šuššum	sixty
šutākulum (~ì-kú (-kú))	to make span
šutamḫurum	to make confront itself
-ta (~*ina*)	from/by
ta-àm	each
tab (Seleucid ~*ṭepû*)	to add
tab (~*eṣēpum*)	to "repeat"
tag$_4$ (~*ezēbum*), cf. íb-tag$_4$	to leave
takīltum	takīltum
takkirtum (cf. kúr)	change
tammar (~igi-dù/du$_8$)	you see
târum (~nigín(-na))	to turn back
tawirtum (~garim)	meadow
tebûm	to go away
TUK, see dug$_4$	
tur (~*ṣeḫrum*)	small
ṭepû (Seleucid ~tab)	to add
u	and
u-gù (~*eli*)	over
ul(a) (~nu)	not
UL.GAR (~*kamārum*)	to accumulate/accumulation
UL.UL	to make encounter
UR.UR	to oppose
uš	length
wabālum	to bring
waṣābum (~daḫ)	to append
wāṣitum/wāṣûm	wāṣitum/wāṣûm
watārum (~dirig)	go beyond
wuṣubbûm	the appended
zi (~*nasāḫum*)	to tear out
zu (~*edûm*)	to know

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JENS HØYRUP

Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought II*

Til Sara og Janne

V. The discourse: Basic second-degree procedures

As stated in section III, the discursive level of Old Babylonian algebra can only be discussed on the basis of actual instances of this discourse. In the present and the following chapters, I shall therefore present a number of texts, translated according to the principle of "conformity" in order to map the original discourse as precisely as possible if the material is not to be presented in the original language. Direct linguistic and philological commentaries are given as notes immediately below the translation of the single texts.

I do not aim at complete coverage of Old Babylonian mathematics. Most practical applications fall outside the scope of the article, and so do the table texts. The application of the specific methods of Old Babylonian algebra to genuine geometric problems are left aside for later treatment, as are most of the "complex" algebraic applications of the basic techniques.[92] Finally, with a single exception only procedure texts are taken into account: Texts which give nothing but the statement of a problem or a series of such statements give little information as long as our understanding of concepts and terminology remains at the present level.

On the other hand, in relation to the class of simple "length-width"-procedure texts the coverage can be regarded as fairly representative. Truly, each text taken into account brings some new information; still, what is left out appears to me to belong to the category of details and shades, which may await subsequent investigation. The basic features of Old Babylonian elementary "length-width-algebra" can, I hope (and think), be presented adequately on the basis of the present selection of texts.

V.1. YBC 6967 (MCT, 129)

The problem deals with a pair of numbers belonging together in the table of reciprocals, the *igûm* and the *igibûm*. The Sumerian forms igi and igi-bi mean „the igi" and "its igi"; they are used most of the way through the text, but a

* For the first part see p. 27–69 of this volume.

[92] I discuss the problems of two-dimensional geometrical conceptualizations and methods and a number of complex algebra problems in my preliminary (1985: 41–63, 105.1 to 105.42).

syllabic *i-gu-um* in rev. 5 indicates that the terms are to be read as Akkadianized loanwords though mostly written logographically.[93] Their product (the "surface" of obv. 9) is supposed to be 1` (=60), or at least an odd power of 60, not 1°. In conformal translation and transliteration, the text runs as follows (to facilitate mathematical understanding, the left margin gives a totally anachronistic commentary in symbolic algebra – *igibûm* $=x$, *igûm* $=y$):

Obverse

$[x\cdot y=60,]\ x-y=7$	1. The igibûm *over* the igûm 7 *goes beyond* [igi-b]i *eli* igi 7 *i-ter*
$x?\ \ y?$	2. igûm *and* igibûm *what?* [igi] *ù* igi-bi *mi-nu-um*
	3. *You*, 7 *which* the igibûm *a*[*t-t*]*a* 7 *ša* igi-bi
	4. over the igûm *goes beyond* ugu igi *i-te-ru*
$\frac{x-y}{2}=3^1/_2$	5. *to two break:* 3° 30′. *a-na ši-na ḫi-pí-ma* 3,30
$\left(\frac{x-y}{2}\right)^2=12^1/_4$	6. 3° 30′ *together with* 3° 30′ 3,30 *it-ti* 3,30
	7. *make span:* 12° 15′ *šu-ta-ki-il-ma* 12,15
$\left(\frac{x-y}{2}\right)^2+x\cdot y=$	8. *To* 12° 15′ *which comes up for you* *a-na* 12, 15 *ša i-li*⟨*-a*⟩*-kum*
$\left(\frac{x+y}{2}\right)^2=72^1/_4$	9. 1` the sur*face append:* 1` 12° 15′. [1 *a*-šà-*l*]*a-am ṣí-ib-ma* 1, 12, 15
$\frac{x+y}{2}=\sqrt{72^1/_4}=8^1/_2$	10. The equilateral of 1` 12° 15′ *what?* 8° 30′. [íb-si$_8$ 1], 12, 15 *mi-nu-um* 8, 30
	11. 8° 30′ *and* 8° 30′ *its counterpart lay down:* [8, 30 *ù*] 8, 30 *me-ḫe-er-šu i-di-ma*

Reverse

	1. 3° 30′ the *takīltum* 3, 30 *ta-ki-il-tam*
$\frac{x+y}{2}-\frac{x-y}{2}=8^1/_2-3^1/_2$	2. *from* the *one tear out* *i-na iš-te-en ù-su-uḫ*

[93] Another text dealing with *igûm* and *igibûm* is VAT 8520 (MKT I 346f.). There, the names of the two unknowns are written syllabically throughout the tablet, while "part" and "reciprocal" are referred to by the usual ideogram igi. This leaves little doubt that the two ideas were, and thus have to be, kept apart, if not in spoken language then at least as concepts.

$\frac{x+y}{2}+\frac{x-y}{2}=8^1/_2+3^1/_2$	3. *to* the *other append* *a-na iš-te-en ṣi-ib*
$8^1/_2+3^1/_2=12$ $8^1/_2-3^1/_2=5$	4. The *first* is 12, the *second* is 5. *iš-te-en* 12 *ša-nu-um* 5
$x=12,\ y=5$	5. 12 is the igibûm, 5 is the *igûm*. 12 igi-bi 5 *i-gu-um*

If "going beyond" is interpreted as arithmetical difference, "breaking" as arithmetical halving, "making span" as arithmetical multiplication, "surface" as arithmetical product, "equilateral" as arithmetical square root, and *takīltum* as a factor (in agreement with the interpretation "that which is made span"), most of this text could agree with an arithmetical interpretation of Old Babylonian algebra. A few points remain, however, which always have been seen as peculiar. Why is the "counterpart" of the square-root introduced? And why are these two copies of the number 8° 30′ kept so strictly apart as a "first" and a "second" in rev. 2–4?

If a naive-geometric interpretation of the procedure is made, these two questions are immediately solved (cf. Fig. 4): Since the product of *igûm* (y) and *igibûm* (x) is spoken of as a surface, they are to be regarded as width and length of a rectangle. That amount by which the length "goes beyond" the width is bisected together with the adjacent part of the rectangle, and the outer half is moved to a position where it "spans" a rectangle (actually a square) together with the inner half. The area of the resulting gnomon is still 1`. When it is appended to the square spanned by the two halves (of area $(3°\,30')^2=12°\,15'$), we get a greater square of area 1` 12° 15′. The side producing this square, or, rather, as we shall see below, the side produced by the area when the latter is understood as a square figure and thus identified with its side, is $\sqrt{1`\ 12°\ 15'}=8°\ 30'$. It is "laid down" (possibly "drawn", cf. section IV.8) together with its "counterpart" (heavy lines). When "that which was made span" the small square (the *takīltum*) is "torn out" from the vertical heavy line (its secondary position) we get the width (the *igûm*). When it is appended to the horizontal heavy line (its original position) we get the length (the *igibûm*).

It will be noticed that not a single word of the description is superfluous or enigmatic when this interpretation is applied. It can also be noticed that an alternative formulation, the "first" and "second" 3° 30′ appended to and born out from the same 8° 30′ (e.g. the horizontal heavy line) would be less meaningful, producing two lines equal to but not identical with length and width. As it actually stands, the text tells us first to tear out the quantity 3° 30′ in one place and next to append this same quantity, now at our disposal, in another place.[93a]

This sense-making use of "first" and "second" holds throughout the many texts where they are used. That can scarcely be a random phenomenon. So, an

[93a] This (invariable) precedence of the tearing process was observed by Vajman (1961: 100), who also pointed out the implication that the same concrete quantity must be involved in tearing and appending.

In one text translated below the addition comes first, viz. IM 53201 Nº 2 rev. 12f. (section X.1). But precisely in this case the objects of the two operations are the two different "moieties" of an excess. In truth an exception which confirms the rule.

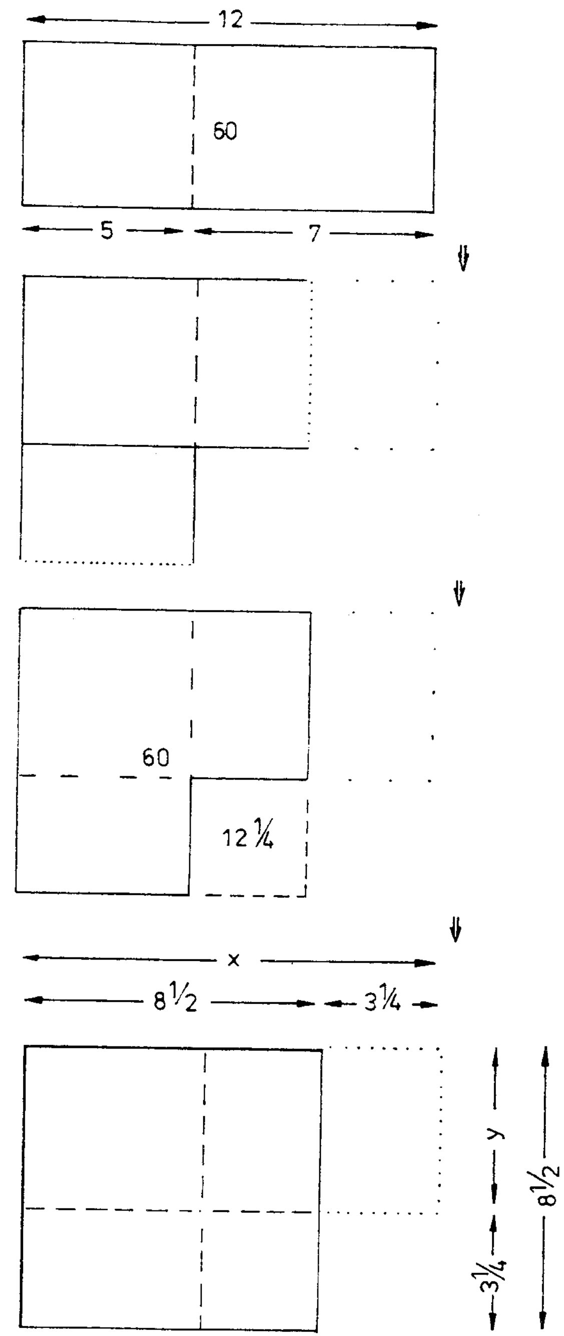


Figure 4. The geometrical interpretation of YBC 6967.

interpretation of the duplication 8° 30′ as nothing but a preparation for two different arithmetical calculations can hardly hold good–in that case, we could expect instances of "first 3° 30′ appended to first 8° 30′, second 3° 30′ born out from second 8° 30′", and other variations of the same sort. In fact, they are never found.

In other respects too, our text is representative of a whole group of procedure texts. As already observed above (section IV.8), the term "to lay down" is always reserved to that process which corresponds to the "drawing of the heavy lines"; if only a number was taken note of for use in an arithmetical calculation, how are we to explain that e.g. the numbers submitted to the operations "appending" and "tearing out" are never "laid down"? Similarly, it is a general feature that 3° 30′ is appended to 8° 30′–that quantity which is moved is appended to that which stays in place. The difference is not one of relative magnitude–as we see in obv. 8f., a greater quantity may well be appended to a smaller quantity; neither is it just a question of fixed habits – when gnomon and square are joined (a situation where both entities are already in place), either can be appended[94]; only where the geometrical interpretation requires that one addend remains in place and one is moved is it apparently impossible to exchange the roles of the two addends. Finally, the concept of a "counterpart" is reserved to roles similar to that which it plays in obv. 11 of the present text; in the case of bisections ("breakings") preparing a purely linear operation it is not used.[95]

As we see, all three features are easily explained inside a geometric interpretation. It is, on the other hand, very difficult to find reasons explaining them if an arithmetical interpretation is taken for granted; and it is extremely improbable that the random selection of surviving sources has created a fixed pattern which did not exist originally–our material is not that small.

It will be observed that the text appears to describe a constructive procedure, not argumentation on a ready-made figure like Fig. 2. It will also be seen that the procedure coincides *grosso modo* with that described by al-Khwārizmī (cf. Fig. 1, AoF 17 [1990], 36).

V.2. BM 13901, Nº 1 (MKT III,1: cf. TMB, 1)

BM 13901 contains a series of problems dealing with one or more squares. The first of these is a precise analogon to the one quoted in Chapter I from al-Khwārizmī. It runs as follows:

Obverse I

$x^2+x={}^3/_4=45'$	1. The sur*face and my confrontation I have accumulated:* 45′. 1 the *wāṣītum*[a] a-šà[l][am] ù *mi-it-ḫar-ti ak-m*[*ur-m*]*a* 45-e 1 *wa-ṣi-tam*

[94] In various problems from BM 13901 (below), the supplementary square is appended to the gnomon; in VAT 8520, as in the present text, the gnomon is appended.

[95] The sole exception from this general rule is IM 52301 (obv. 12, rev. 10). This is only one of several reasons to regard this late tablet as a symptom of changing conceptualizations (cf. below, note 113, section X.1, and note 176).

${}^1/_2 \cdot 1=30'$, $(30')^2=15'$	2. *you pose.* The *moiety*[b] *of* 1 *you break,* 30′ *and* 30′ *you make span,* *ta-ša-ka-an bu-ma-at* 1 *te-ḫi-pi* 30 *ù* 30 *tu-uš-ta-kal*
$x^2+2\cdot 30'\cdot x+(30')^2=15'+45'=1$ $x+30'=\sqrt{1}=1$	3. 15′ *to* 45′ *you append:* 1 makes 1 equilateral[c]. 30′ *which you have made span* 15 *a-na* 45 *tu-ṣa-ab-ma* 1-[e] 1 íb-si$_8$ 30 *ša tu-uš-ta-ki-lu*
$x=1-30'=30'$	4. *in* the *inside of*[d] 1 *you tear out:* 30′ the *confrontation.* *lìb-ba* 1 *ta-na-sà-aḫ-ma* 30 *mi-it-ḫar-tum*

[a] *wāṣītum* is a nominal derivation from *waṣûm*, "herausgehen, fortgehen ... herauswachsen ... hervortreten, herausragen". The term itself means something going out, including something projecting from a building. Since the mathematical application of the term has never been explained before, I have left it untranslated.

[b] The use of a term for a "wing", a "natural" instead of a mere arithmetical half is noteworthy.

[c] "1 makes 1 equilateral" translates "1-e 1 íb-si$_8$". The use of the "agentive suffix" -e (which occurs commonly in this connexion) appears to indicate not only that the verbal character of the term íb-si$_8$ is still present to the Old Babylonian calculator, but also that the first "1" is considered the agent of a transitive verb, while the second "1" must be seen as the object. Cf. Thureau-Dangin 1936a: 31 note 3, which also quotes an instance of the phrase *mi-nam* íb-si$_8$ where a square-root is asked for; here, too, the square-root must be the object of an act since it is asked for in the accusative. (So also the Susa and most Tell Ḥarmal texts).

A number of other texts, however, ask for the square-root by the phrase íb-si$_8$ x *mi-nu-um* (e.g. YBC 6967, obv. 10) or íb-si$_8$ x en-nam (e.g. VAT 8390, passim, and VAT 8520, obv. 20, rev. 19). *mi-nu-um* is an indubitable nominative; in the latter texts, the other occurrences of en-nam are indubitable nominatives, while corresponding accusatives are written phonetically as *mi-nam.* In such cases (and when the term is used in the generalized sense of "solution" to an equation), íb-si$_8$ must apparently be read as a noun, and I shall translate "the equilateral of x how much".

In a few late OB and in one early northern text, the alternative term ba-si$_8$, originally a verb too, has been adopted into Akkadian as a loanword *basûm*, which is regarded completely as a boun – cf. IM 52301, No 2, note d (below, section X.1).

[d] Thureau-Dangin (1936a: 31 note 4) explains the form *lìb-ba* (ŠÀ.BA) as *libba*, the construct state of a locativic accusative. Another possible interpretation reads ŠÀ = sag$_4$ ~ *libbum*, BA = ba < bi-a, compound possessive + locativic suffix (cf. SLa § 182).

We observe that the "confrontation" is in fact identical with the side of the square, while the area of that figure is spoken of by a separate concept, "the surface".

When this usage is accepted, the procedure is *grosso modo* mapped by the arithmetico-symbolic interpretation in the left margin. However, it remains fully unclear why the number 1 should be spoken of as something "projecting" or "going away". Another puzzle is the choice of the term *bāmtum*, "moiety", when the normal term *mišlum*, "half", is used everywhere in the tablet when one entity is the half of another entity.

If we try a geometric interpretation, the intention of both terms can be made clear (see Fig. 5).

As in al-Khwārizmī, a geometric summation of a square and a number a of sides requires that the number a is understood as having the dimension of a length. This is shown in the first step of the figure, where the "confrontation" is represented by the area of a rectangle of length *1* and width x. The figure makes it immediately obvious that the number *1* is something which projects. The only question which is left open is whether it projects from the square or from the width x[96] (as we shall see below, the latter possibility must be preferred).

From here, the procedure is exactly parallel to that of YBC 6967 and Figure 4. Comparing the two texts we can even see why the need for the term *wāṣitum* arises: while the problem of two unknowns could speak of that by which "x goes beyond y", the corresponding geometrical quantity *1* ("that by which $x+1$ goes beyond x") has no obvious designation in the problem of one unknown – if not, precisely, *wāṣitum*. This is then posed and next "broken" (i.e. bisected), and the outer half is moved so that a square is spanned. This square is appended to the gnomon resulting from the preceding manipulations of the figure, in order to produce another square. The side of this great square is found (literally: the result 1 of the appension produces 1 as "equilateral"). Finally, the quantity which spanned the complementary square[97] is removed ("torn out"), and the unknown side of the original square (the original "confrontation") is left.

Concerning the "moiety", the situation in the figure is evidently related to the origin of the term. By the very nature of the problem, the appended rectangle consists of two "wings", of which one is to be broken off and moved.

According to both F. Thureau-Dangin and O. Neugebauer, the tablet belongs

[96] In its own way, this confirms O. Neugebauer's old intuition. F. Thureau-Dangin suggested very tentatively (1936a: 31 n.1) that *wāṣitum* might designate absolute unity as distinct from 1′, 1` etc.). Against this, O. Neugebauer (MKT III 11) raised the objection that only absolute unities belonging with problems of one unknown were designated *wāṣitum*. Instead, he suggested that the term might designate a certain class of coefficients of value *1*. Irrespective of the precise interpretation, indeed, the "projection" is a coefficient *1* of dimension (length), multiplication by which transforms a linear quantity into a quantity of dimension [length^2].

[97] In YBC 6967, this quantity was spoken of by the noun *takīltum*, here however by the relative clause "which you have made span", *ša tuštakkilū*. This parallel (which is repeated copiously) confirms the close relation between "making span" (*šutākulum*) and *takīltum*.

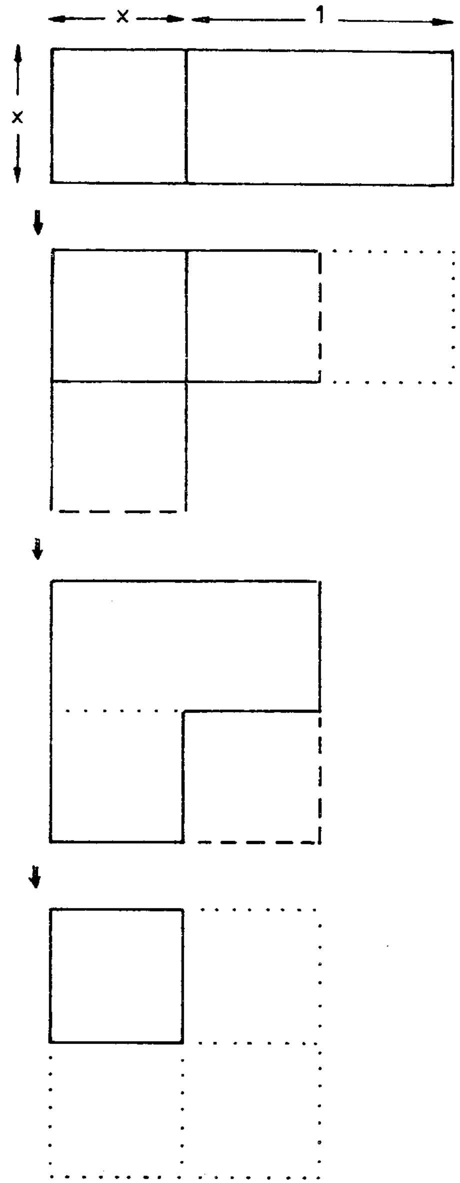


Figure 5. The geometrical interpretation of BM 13901 N° 1.

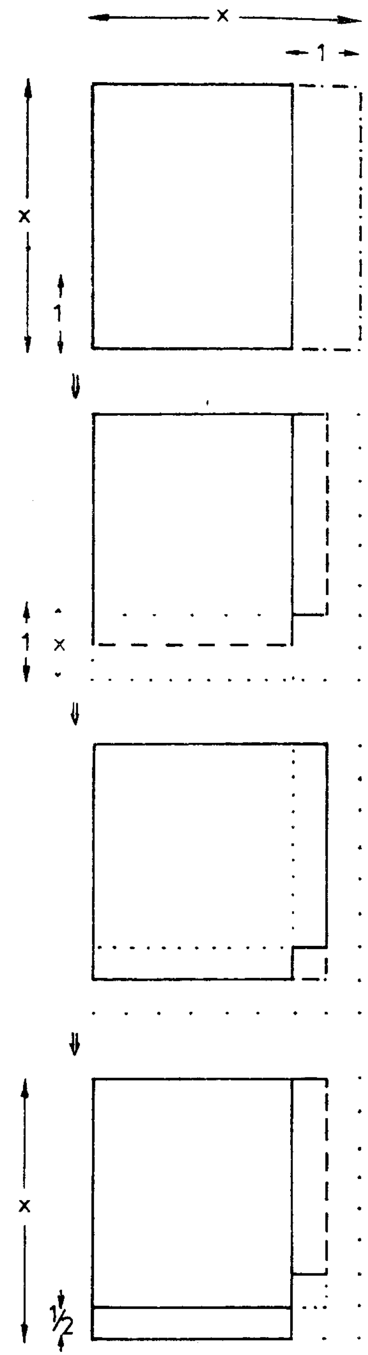


Figure 6. The geometrical interpretation of BM 13901 N° 2 (distorted proportions).

together with AO 8862 to the oldest stratum of Old Babylonian mathematics.[98] A. Goetze's linguistic analysis ascribes to both a southern origin, probably Larsa.[99]

V.3. BM 13901, N° 2 (MKT III, 1; cf. TMB, 1)

The second problem of the tablet subtracts a side instead of adding it. The text runs as follows:

Obverse I

$x^2-x=14^{\backprime}\,30^\circ$	5. *My confrontation inside of* the surface *I have torn out:* 14‘ 30°. 1 the *wāṣītum* *mi-it-ḫar-ti lìb-bi* a-šà *[a]s-sú-uḫ-ma* 14, 30 1 *wa-ṣi-tam*
$^1/_2 \cdot 1 = 30'$ $(30')^2 = 15'$	6. *you pose.* The *moiety of* 1 *you break,* 30′ *and* 30′ *you make span;* *ta-ša-ka-an ba-ma-at* 1 *te-ḫi-pi* 30 *ù* 30 *tu-uš-ta-kal*
$x^2-2\cdot 30'\cdot x+(30')^2=$ $14^{\backprime}\,30^\circ\,15'$ $x-30'=\sqrt{14^{\backprime}\,30^\circ\,15'}=$ $29^\circ\,30'$	7. 15′ *to* 14‘ 30° *you append:* 14‘ 30° 15′ makes 29° 30′ equilateral. 15′ *a-⌜na* 14, 30 *tu-ṣa-]ab-ma* 14, 30, 15-e 29, 30 íb-si$_8$
$x=29^\circ\,30'+30'=30^\circ$	8. 30′ *which you have made span to* 29‘ 30° *you append;* 30 the *confrontation.* 30 *ša tu-uš-ta-ki-lu a-na* 29, 30 *tu-ṣa-ab-ma* 30 *mi-it-ḫar-tum*

Once again, the text is *grosso modo* mapped by the arithmetico-symbolic interpretation. Only the problem of the "1 which projects" is left open, together with the question why only the "coefficient" of the first-degree term is "posed", and the choice of the term "moiety".

If the imagery inherent in the terminology ("appending", "tearing out", "breaking", "making span") is taken at face value, we are led to a geometric procedure which solves even these problems (see Fig. 6). From the square, a rectangle of length x and width *1* is removed. The area of the remaining rectangle is 14‘ 30°. Since the length of this rectangle exceeds the width by *1*, a strip of this width is bisected, and its outer wing is moved so as to transform the known area into a gnomon. The small square spanned by the two halves of the strip is appended, and so we get a square of known area. Its side is found, and the half-strip which was moved in order to span the small square is appended again. This gives us the original length of the rectangle, and thus the side x of the square.

The geometrical procedure is of course the same as that of Fig. 4 and Fig. 5: The area of a rectangle is given, together with the difference between its length and its width. The excess of length over width is bisected, and the rectangle is transformed into a gnomon, for which the area and the side of the lacking square are known. The area of the lacking square is then found and added to the gnomon,

[98] Thureau-Dangin 1936a: 27; MKT III 10. The criteria are language and writing.
[99] In MCT 148, 151.

transforming it into a square of known area. The side of this square is calculated, and the original length (Fig. 6), width (Fig. 5) or both (Fig. 4) can finally be found. Indeed, the only difference between the cases (as seen from the geometrical interpretation) concerns the entity asked for.

It is still not to be seen whether the *wāṣītum* should be understood as that width *1* which must project from the length in order to transform it into an area which can be torn out, or perhaps as the excess of rectangular length over rectangular width. In any case, it has a definite role to play in the procedure (and as stated above, the former possibility will turn out to be correct). In the geometric interpretation the question thus disappears why only the coefficient *1* to the linear term is posed—the *wāṣītum* is no numerical coefficient.

Once again, then, the aritmetic-algebraic interpretation allows us to understand the main mathematical progress of the calculation but not the details of the formulation; the approach through naive geometry, on the other hand, allows us to understand both the mathematical progress and the discursive organization of the texts.

V.4. BM 13901, N° 23 (MKT III, 4f.; cf. TMB, 17f.)

The three previous problems presented the standard way to solve the basic mixed second-degree equations. The present one exemplifies that the Babylonians would sometimes leave the standard methods.

The problem adds the four sides of a square to the surface – not 4 times the side, but explicitly *the four sides*:

Reverse II

$x^2+4\cdot x=41'\,40''$	11. *In* a surface, *the four fronts and* the surface[a] *I have accumulated:* 41′ 40″ a-šà^{lam} *p[a]-a[-at er-bé-et-tam ù* a-š] à^{lam} *ak-mur-ma* 41, 40
	12. 4, *the four fronts, you inscribe.* The igi of 4 is 15′ . 4 *pa-a-at er[-bé-e]t-tam t[a-la-p]a-at* igi 4 gál-bi 15
$^1/_4 x^2+x=10'\,25''$	13. 15′ *to* 41′ 40″ *you raise:* 10′ 25″ *you inscribe.* 15 *a-na* 41, 40 *[ta-n]a-ši-ma* 10, 25 *ta-la-pa-at*
$(^1/_2 x+1)^2=10'\,25''+1$ $=1^\circ\,10'\,25''$ $^1/_2 x+1=\sqrt{1^\circ 10' 25''}=1^\circ 5'$	14. 1 the *wāṣītum you append:* 1° 10′ 25″ makes 1° 5′ equilateral 1 *wa-ṣi-tam tu-ṣa-ab-ma* 1, 10, 25-e 1, 5 íb-si$_8$
$^1/_2 x=1^\circ\,5'-1=5'$	15. 1 the *wāṣītum which you have appended* *you tear out:* 5′ *to two* 1 *wa-ṣi-tam ša tu-iṣ-bu ta-na-sà-aḫ-ma* 5 *a-na ši-na*
$x=2\cdot 5'=10'$	16. *you repeat:* 10′ nindan *confronts itself*[b] *te-ṣi-ip-ma* 10 nindan *im-ta-ḫa-ar*

[a] This passage is very unusual, indeed without parallel in mathematical texts, and thus of special interest. First there is the initial statement that we are dealing with a surface. In itself, the use of an accusative form here is not impossible; most plausibly it is to be interpreted as an locativic accusative (cf. GAG § 146). However, in other cases where the subject of a problem is stated this is always done by Sumerograms without any complement (uš sag e.g. in AO 8862, túl-sag in BM 85200 + VAT 6599). In the present case, it seems to be important to stress either the use of an accusative form or the specific Akkadian pronunciation—even though the whole tablet is dominated by syllabic writing, complements are attached to a-šà only when they are needed to impede misunderstanding. The use of pure Sumerograms in parallel texts indicate that there was no general need to display an accusative case explicitly; most probably, then, the complements are meant to indicate the use of an Akkadian archaism.

The "fronts" translate *pāt*, plural (construct state) of *pūtum*. This word is often considered an equivalent of sag, my standard translation of which is "width". Only extremely few texts, however, use the Akkadian word instead of the Sumerogram, and none of them belong to the category of standard "length and width"-problems (see above, note 75). Even occurrences of the Sumerogram with an Akkadian phonetic complement are strictly absent. The use of the term *pūtum* in our text must thus intend something explicitly different from the technical concept "width"—perhaps another archaism. Hence my use of the literal translation "front".

The numeral "four" is in status rectus and postponed. This literary stylistic figure appears to belong to situations where the number is an invariable epithet, i.e. where n items belong invariably together ("the seven mountains", cf. GAG § 139i), whence "*the four*" instead of "the *four*".

[b] The term is *imtaḫḫar* (or possibly *imtaḫar*, the preterite form), Gt-stem of *maḫārum*, "to confront".

This time, the arithmetico-algebraic interpretations lead into real trouble. Indeed, if a "square" is only a second power, there is no reason to speak of *the four* fronts (or widths); neither is there any reason to leave the normal concept of the "confrontation" for that of "front", nor to specify in this case alone that we are dealing with a "surface".

Of course, an arithmetical interpretation can map the mathematical procedure. But it offers no explanation why normal terminology and procedure are given up in this specific case; in fact, the deviation is so astonishing that O. Neugebauer suspected it to have arisen by a combination of mistakes which happen to make sense.[100] Finally, the place of the problem on the tablet (among the complicated variations and not among the simple cases of one variable) is an enigma; so is also the "repetition to two" in a place where an arithmetic interpretation would expect a "raising" (cf. the problem discussed immediately below.)

The geometric interpretation, especially as it is made clear by the term *wāṣitum*, solves many of these problems (cf. Fig. 7). First of all it is clear that a geometric

[100] MKT III 14.

square possesses four sides, which can be regarded as "fronts". Moreover, if we take the text at its words and add four rectangles of length *1* and width x instead of one rectangle of length *4* and width x, or two of dimensions *2* times x, as we would normally expect), we get a geometrical configuration which differs from the normal square-plus-sides dealt with in the beginning of the tablet—and thus a reason that the problem is listed among the complicated variations.

The occurrences of the *wāṣitum* confirm that the cross-form configuration is indeed thought of: If we follow the text, we can imagine the multiplication by $1/4$ in lines 12f. as a quartering, as shown in the second step on the figure. At first, this is of course only a possibility. In line 14, however, the *wāṣitum* is appended, i.e., not any number *1* but a square 1^2 identified with the *wāṣitum*; such a square is shown in the third step[101], where it completes the quartered cross as a square. No other configuration than the cross would allow so literal a reading of the text, — and since the occurrence of the *wāṣitum* in line 14 does not refer to any earlier occurrence, it must refer to the entity itself, not to anything obtained from or equal to the "projection".

In the next step of line 14, the side of the completed square is found, and in line 15 the same *wāṣitum* is torn out. This rules out F. Thureau-Dangin's conjecture, viz. that the term may simply fix the order of magnitude to 1° (one need not fix the order of magnitude of a number which is identical with a number previously used), and it confirms that the square which was appended in line 14 is identified with its side: if a squaring of *1* had been left out by error in line 14, the invariable epithet would have been "which you have made span" instead of "which you have appended" (cf. problems N° 1 and 2 from the tablet as quoted above).

The tearing-out of the *wāṣitum* leaves half the side of the square (in the right position). It is "repeated to two", and indeed repeated quite concretely[102], in agreement with the situation of the figure, giving us one of the fronts. It is, however, not spoken of as a "front", nor designated by the normal term "confrontation" (*mitḫartum*). Instead, it is stated that 10′ is that which "confronts itself" — presumably because no "confrontation" was spoken of explicitly in the statement of the problem; instead four "fronts" have been supposed to "confront each other as equals".

Curiously enough, al-Khwārizmī uses the same figure as an alternative argument for the solution of the problem "square and roots equal to number" (cf. above, section 1). Here, instead of distributing the rectangle $10 \cdot x$ as shown

[101] We notice that the current identification of a square with its side can explain that the *wāṣitum* itself is appended, and not a "1" spanned by the *wāṣitum* together with itself. At the same time we observe that the entity which is "appended" must be the concrete geometric piece of surface, not a number measuring its magnitude: Such a number would, even to the Babylonians, have to be found via one of the "multiplicatory" processes "making span" or "raising", as are all "surfaces". Due to the configuration, however, there is no need to "make the *wāṣitum* span", i.e. to make it form a rectangle (in fact a square): The square is already there, spanned by the corner of the cross—there is no need to prescribe its construction.

[102] The specification "to two" shows that the original sense of *eṣēpum* (to duplicate, i.e. to repeat once) has been absorbed into the generalization "to repeat N times". Genuine duplication has been left behind.

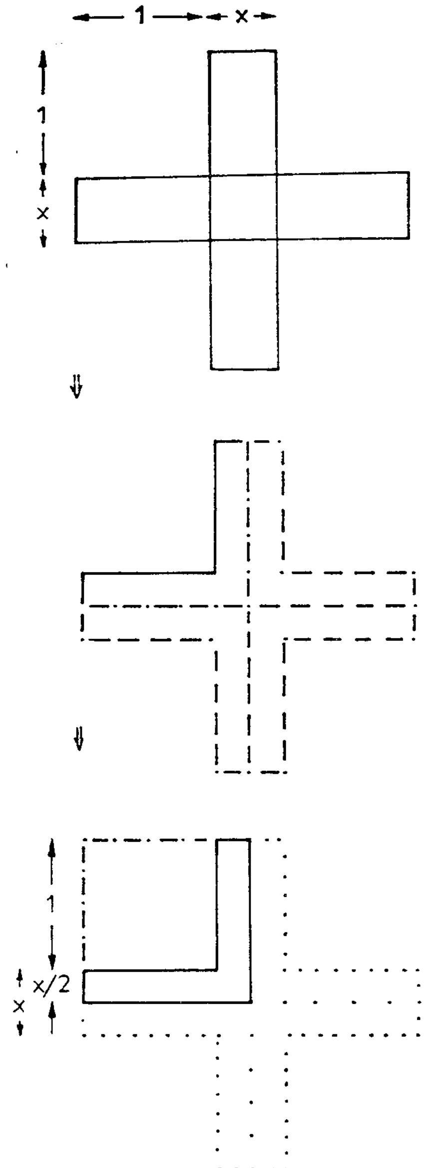


Figure 7. The geometrical interpretation of BM 13901 N° 23.

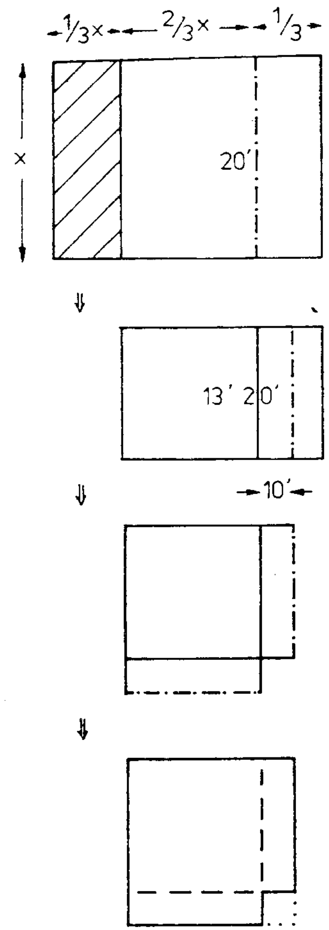


Fig. 8. The geometrical interpretation of BM 13901 N° 3.

in Fig. 1, he distributes it as four rectangles $2^1/_2 \cdot x$ along the four edges of the square.[103]

[103] See Rosen 1831: 13–15. It cannot be decided on the basis of the al-Khwārizmī-text and the present Old Babylonian text alone whether the recurrence of two Old Babylonian methods in al-Khwārizmī's *Algebra* is due to coincidence or to continuous tra-

The text brings us somewhat closer to the precise meaning of the *wāṣītum*. It cannot be the excess of rectangular length over rectangular width. Possibly, it could be the length of any of the four projections from the central square; that would, however, agree poorly with the use in problem 2 of the tablet (see above). So, we are led towards the interpretation of the *wāṣītum* as that projecting width *1* which transforms a length into a rectangle of equal area.

One problem in the text is not elucidated by the naive-geometric interpretation in itself, viz. the initial "in a surface" or "concerning a surface".[103a] If nothing but the area of an ordinary square is meant, this indication is superfluous, and not to be expected after 22 problems which all deal with square areas without mentioning them explicitly beforehand. Together with other evidence which will be presented in section X.4, however, the apparent archaisms of the language may offer an explanation: *eqlum* is not only a (semi-)technical term for a mathematical area but also the everyday term for a field. All evidence combined suggests that the problem is a surveyors' recreational problem, maybe from a tradition which was older than—perhaps even a source for—Old Babylonian scribal school "algebra". The initial *eqlam* can be understood as an indication that we are dealing with a field-surveying problem (albeit an artificial one), and the apparent archaism perhaps as a reference to age and tradition or perhaps to oral or dialect usage (locativic and similar accusatives are more common and long-lived in Assyrian than in Babylonian).

V.5. BM 13901, N° 3 (MKT III, 1; cf. TMB, 1f.)

The above problems can all be classified as "normalized mixed second-degree equations". The present problem shows the habitual Old Babylonian way to deal with a non-normalized equation. The text runs as follows:

Obverse I

$(1-{}^1/_3)x^2+{}^1/_3x=20'$ 9. The *third of* the s u r f a c e *I have torn out:* the *third of* the *confrontation to* the *inside*
ša-lu-uš-ti a-šà *as-sú*⟨*-uḫ-ma*⟩ *ša-lu-uš-ti mi-it-ḫar-tim a-na lìb-bi*
10. *of* the sur*face I have appended:* 20′. 1 the *wāṣītum you pose*
a -šà*lim* *ú-ṣi-ib-ma* 20-e 1 *wa-ṣi-tam ta-ša-ka-an*
$1-{}^1/_3=1°-20'=40'$ 11. The *third of* 1 the *wāṣītum*, 20′ *you tear out:* 40′ *to*

dition. As I shall show in section X.4, however, another algebraic text roughly contemporary with al-Khwārizmī's shows continuity with the Old Babylonian tradition even down to the choice of grammatical forms, while displaying the same interest as the present problem in the four sides of squares and rectangles. This leaves little doubt that al-Khwārizmī too was inspired by the same old tradition.

[103a] Initially I believed so, reading the text as "The surface of the four fronts and the surface I have accumulated . . .", interpreting the "surface of the four fronts" as the total surface of the "arms" of the cross. I am grateful to A. Westenholz for pointing out the grammatical objections to this reading.

$(40'x)^2+40'\cdot 20'x = 13'\,20''$		20′ *you raise;* *ša-lu-uš-ti* 1 *wa-ṣi[-tim* 20 *ta-na-sà-aḫ-ma]* 40 *a-na* 20 *ta-na-ši*
$(40'x)^2+2\cdot 10'\cdot(40'x)+(10')^2=$ $13'\,20''+1'\,40''=15'$	12.	13′ 20″ *you inscribe*. The *moiety of* 20′, the *third which you have torn out*[a] 13, 20 *ta-la-pa-at [ba-ma-at* 20 *ša-l]u-uš-tim ša ta-sú-ḫu*
$(40'x+10')^2=15'$	13.	*you break:* 10′ *and* 10′ *you make span,* 1′ 40″ *to* 13′ 20″ *you append* *te-ḫi-pi* 10 [*ù* 10 *tu-uš-ta-kal* 1, 40] *a-na* 13, 20 *tu-ṣa-ab*
$40'x+10'=\sqrt{15'}=30'$ $40'x=30'-10'=20'$	14.	15′ makes 30′ equilateral. 10′ *which you have made span in* the *inside of* 30′ *you tear out:* 20′. 15-e 30 [ib-si$_8$ 10 *ša tu-uš-ta-ki-lu lìb-ba* 30] *ta-na-sà-aḫ-ma* 20
$(40)^{-1}=1°\,30'$ $x=1°\,30'\cdot 20'=30'$	15.	The igi of 40′, 1° 30′ *to* 20′ *you raise:* 30′ the *confrontation.* igi 40 gál-b[i 1, 30 *a-na* 20 *ta-na-ši-ma* 30] *mi-it-ḫar-tum*

[a] Both for mathematical reasons and because of the many parallel passages of the tablet, this "have torn out" must be a writing error for "you have appended", *tu-iṣ-bu*.

The problem is of the type $\alpha x^2+\beta x=\gamma$. In Medieval (Arabic and Latin) algebra, such an equation would be normalized as $x^2+(\beta/\alpha)x=(\gamma/\alpha)$. The method here is different, a fact which has often been regarded as astonishing, although the same procedure is used by Diophantos and Hero[104]: Instead of x, αx is taken as the quantity looked for, and the equation is transformed into $(\alpha x)^2+\beta\cdot(\alpha x)=\alpha\gamma$. In the end, x is found from αx through multiplication by the reciprocal of α.

The application of the arithmetical interpretation raises a problem: The multiplications by α and α^{-1} are expressed by means of the term "to raise", while that of $(\beta/2)$ by $(\beta/2)$ (of 10′ by 10′) is expressed by "making span". Another problem is presented through the way the equation is transformed: As most of us would immediately feel, and as it is confirmed by the Medieval algebras, in a rhetorico-arithmetic representation it is easier to keep track of a reduction to normalized form that of the actual "change of variable". Finally, of course, the *wāṣītum* remains a stranger to any arithmetical interpretation, as does the distinction of a "moiety" from a "half".

As usual, we shall try to apply a representation by naive geometry—see Fig. 8. If we look at lines 12–14 of the text, it is clear that they follow the normal "square-plus-sides"-procedure (cf. section V.1 and Fig. 5). So, we must interpret the text geometrically in such a way that this situations comes about.

[104] Diophantos, *Arithmetica* VI, vi. Hero, *Geometrica* 21, 9f. The Diophantine and Heronian parallels have been pointed out by K. Vogel (1936: 714; 1959: 49).

Line 9f. states the problem. In line 10, furthermore, the *wāṣītum* is "posed", and since no "projection" from the square is 1, we can now be sure that the term designates that projection from a line which creates the rectangle of equal area, as suggested above. An area of one third of the side is then a rectangle of width "the third of *1* the *wāṣītum*", i.e. 20′, and length x. This corresponds to line 11 where, however, an ellipsis turns up, as the third of the *wāṣītum* is identified with that third (of the surface) which is to be "torn out"; that such a confusion is really there is confirmed in line 12. So, the "coefficient to x^2" (α) is found to be $1°-20'=40'$.

In the last part of line 11, this factor is applied to the total non-shaded area ($^2/_3x^2+{}^1/_3x=20'$). This can be apprehended geometrically as the first transformation of the figure, where the scale factor 40′ ($={}^2/_3$) is applied in the vertical direction. This operation transforms the rectangle $x\cdot{}^2/_3x$ into a square $^2/_3x\cdot{}^2/_3x$. In the same process, the appended rectangle $^1/_3\cdot x$ is transformed into a rectangle $^1/_3\cdot{}^2/_3x$. That is, we have obtained the required situation "square-plus-sides", and the number of "sides" is unchanged. The rest of the procedure is by now well-known: The appended rectangle is bisected and its outer wing moved so as to "span" a square of area 1′ 40″. This area is appended to the gnomon, the area of which is $40'\cdot 20'=13'\,20''$. The area of the resulting square is 15′, and its side therefore 30′. From this, the side 10′ of the square which was "spanned" is "torn out", leaving 20′ as the side of the square $(\alpha x)^2$. Hence, αx is 20′ and x itself is found through division by the scale factor 40′, i.e. through multiplication by its inverse 1° 30′, to be 30′.

This solves all the problems raised by the arithmetical interpretation. First of all, it is clear that the multiplication by a scaling factor or its inverse is different from the geometrical process "to span a square". If the conceptualization and method of Old Babylonian algebra are geometric, a terminological distinction between the two is next to obligatory.

Next, the geometrical interpretation leads us to prefer the "Diophantine" to the "Medieval" reduction: If the non-shaded part were to be transformed into a "square-plus-sides" through Medieval reduction, the change of scale would have to be in the horizontal direction. This would affect the width of the appended rectangle, which goes into the further calculations; on the other hand, the "Diophantine" transformations affects only its length which is anyhow irrelevant.[105]

[105] This simplification of the geometrical prodecure is not in general accompanied by calculatory simplification: The multiplication $\beta\cdot\alpha^{-1}$ is dispensed with, it is true; but the final inverse scaling would be dispensed with in the "Medieval" reduction. Only cases where α is an irregular which does not divide β and γ would be harder—indeed impossible—to deal with "Medievally".

Of course, such arguments of conceptual simplicity should be used with care. We cannot conclude in that way that Diophantos made use of geometric representations. By his syncopated rhetorics he could keep track of problems much more complicated than the present one. But the Babylonian texts were made neither by nor for mathematicians of Diophantine stature; they were school texts, made for scribe students, comparable in giftedness and interests to the students of Medieval merchant ("abacus") schools, we may guess. If the latter were unable to use the Diophantine method in a rhetorical representation, there is no reason to believe that Babylonian students were any better off.

Finally, of course, the *wāṣītum* is no stranger but a must for a geometrical interpretation (with or without a name), and the "moiety" is a natural half, a "wing".

On the other hand, the geometrical interpretation raises two new questions. The first of these concerns the semantic range of the term "raising": Is it restricted to multiplications which can be regarded as changes of scale, or is it wider? This cannot be answered from the present text, but as discussed above (section IV.3) the range is indeed much wider. (Cf. also below, section V.8.)

The second question concerns the figure: Did the Babylonians draw or imagine a series of different diagrams, as they are shown in Fig. 8? Or were they able to conceptualize the same representation first as a rectangle with sides x and $^2/_3x$, and next as a square with both sides equal to $^2/_3x$? It is equally impossible to answer this second question on the basis of the present text (or to give a definitive answer on the basis of any text I know). Yet, as I shall argue in chapter VI, indirect evidence suggests that the Babylonians were fully able to conceptualize a drawn rectangle as a diagram for a square.

The geometrical technique which appears to be used in the first examples and in al-Khwārizmī's justification can be described as a "cut-and-paste"-procedure. The same technique is used in the present example for those operations which are described by the terms "to tear out", "to append", "to break" and "to make span". The "raisings" of line 11 and 15, however, belong with another technique, of which special notice should be taken: A technique of proportionality, which in relation to the geometric representation can be described as a uni-directional "change of scale"; I shall use the term "scaling" for the technique.[106]

V.6. BM 13901, N° 10 (MKT III, 2f.; cf. TMB, 4)

The above examples were all concerned with mixed second-degree equations. We shall now turn to homogeneous problems—first to BM 13901 N° 10.

Obverse II

$x^2+y^2=21° \; 15'$	11. The surfaces *of my two confrontations I have accumulated:* 21° 15′. a-šà *ši-ta mi-it-ḫa-ra-ti-ia ak-mur-ma* 21, 15
$y=(1-{}^1/_7)x={}^6/_7x$	12. *confrontation to confrontation*, the *seventh*[a] *it has diminished.* *mi-it-ḫar-tum a-na mi-it-ḫar-tim si-bi-a-tim im-ṭi*
$x=7z \quad y=6z$ $x^2=49z^2$	13. 7 *and* 6 *you inscribe.* 7 *and* 7 *you make span,* 49. 7 *ù* 6 *ta-la-pa-at* 7 *ù* 7 *tu-uš-ta-kal* 49
$y^2=36z^2$ $x^2+y^2=(49+36)z^2=$	14. 6 *and* 6 *you make span,* 36 *and* 49 *you accumulate:* 6 *ù* 6 *tu-uš-ta-kal* 36 *ù* 49 *ta-ka-mar-ma*
$1`\,25°z^2=21°\,15'$	15. 1` 25°. The igi of 1` 25° *is not detached. What to* 1` 25° 1, 25 igi 1, 25 *ú-la ip-pa-ṭa-ar mi-nam a-na* 1, 25

[106] The method is closely related to the method of a "single false position", which was also used by the Babylonians as a purely arithmetical technique (cf. K. Vogel 1960).

$15' \cdot 1`\,25° = 21°\,15'$ $z^2=15',\ z=\sqrt{15'}=30'$	16. *shall I pose which* 21° 15′ *gives me?* 15′ makes 30′ equilateral. *lu-uš-ku-un ša* 21, 15 *i-na-di-nam* 15-e 30 íb-si$_8$
$x=7\cdot 30'=3°\,30'$	17. 30′ *to* 7 *you raise:* 3° 30′ the *first confrontation.* 30 *a-na* 7 *ta-na-ši-ma* 3, 30 *mi-it-ḫar-tum iš-ti-a-at*
$y=6\cdot 30'=3$	18. 30′ *to* 6 *you raise:* 3 the *second confrontation.* 30 *a-na* 6 *ta-na-ši-ma* 3 *mi-it-ḫar-tum ša-ni-tum.*

[a] The form is a plural, *sebiātim*, cf. Thureau-Dangin 1934: 49, and Goetze 1946: 200.

A geometrical interpretation of the procedure is shown in Fig. 9. The first step, that of finding the set of proportionate numbers, looks like a purely arithmetical "single false position": A number from which one seventh is easily taken away is 7, and the removal of the seventh leaves 6.[107] These numbers are "inscribed", an expression which was also used in N° 23 and N° 3, where the areas

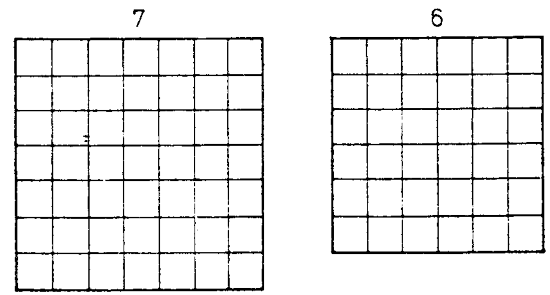


Figure 9. The geometrical interpretation of BM 13901 N° 10.

found by quartering and scaling were "inscribed". In agreement with Babylonian habits as expressed on tablets with drawings[108], we may image inscriptions along the edges of squares, as shown on the figure. The process can be so interpreted that a unit is imagined in which the lengths of the squares are 7 and 6, respectively. Such a conceptualization could follow as an extrapolation from common experience with metrological conversions. The respective areas are found (by "making span") in the square of this unit, as 49 and 36; the total area when measured so will then be $49+36=1`\,25°$. In the basic area unit it is known to be $21°\,15'$. So, the square of the imagined unit (the area of the small squares) is $21°\,15'/1`\,25°=15'$; hence its side will be $\sqrt{15'}=30'$, and those of the two original squares $7\cdot 30'=3°\,30'$ and $6\cdot 30'=3$.

Fundamentally, this conceptualization subdivides the given squares of the problem directly. An alternative interpretation could be that two auxiliary squares are imagined, of "real" sides 7 and 6. Their areas are found and added; the ratio between this and the original total area is calculated, etc.

It is impossible to decide from the text which interpretation to prefer. From

[107] The same pattern of thought is made explicit e.g. in VAT 7532 rev. 6f.
[108] Str. 367 (MKT I 259f.) may be quoted as an example.

the view-point of mathematics, they are of course equivalent.[109] My intuitive feeling is that the former is the more plausible, as it is conceptually simpler—it is easier to draw the subdivisions of an existing square, to point to it and speak about it, than to make non-mathematicians understand an abstract ratio and the reason why its square-root should be taken. As we shall see in the following examples, there is also direct evidence that the Babylonians used subdivisions and alternative "units" rather than ratios·

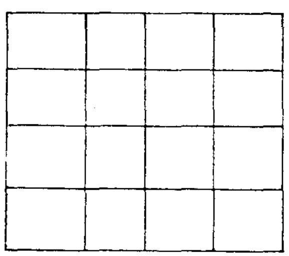
In any case, the text presents us with a third technique besides the cut-and-paste procedures and the scaling: The calculation of total "coefficients"—here the "number of small squares". Below, we shall meet in section VII.3, TMS XVI, the expression "as much as there is of" entity x, as an explicit formulation of this concept.—We notice that the number is found by "accumulation", not by "appending". The same holds for the calculation of the true total area in line 11. In both cases, indeed, none of the addends possesses an "identity" which is conserved through the process. It seems plausible, too, that "accumulation" is a more genuinely arithmetical process than "appending", adding measuring numbers, while "appending" affects only concrete though measured entities.

In order to point to a practice with which the Babylonians were utterly familiar, and which is structurally analogous to the accumulation of a coefficient, I shall speak of the "accounting technique".

V.7. BM 15285, Nº 10 (MKT I, 138; MKT II, Plate 4)

BM 15285 is (part of) a large tablet where the areas of various subdivisions of a square of side 1 are asked for. The present problem is clearly related to a particular aspect of the argument of the previous problem, and it can serve to elucidate the questions left open there.

The text is accompanied by a figure, which I show in the left margin (traced after the photo in MKT II).



1. 1 the length, *a confrontation*
   [1 uš *mi-i*]*t-ḫa-ar-tum*
2. In its inside, 16 *of* a *confrontation*[a]
   sag$_4$-ba 16 *mi-it-ḫa-ar-tim*
3. *I have laid down.* Its surface what?
   ad-di a-šà-bi en-nam

[a] The form is a genitive singular.

The figure shows us precisely the subdivision of a square into smaller squares which was suggested as the first interpretation of the procedure of the previous problem. So, this interpretation is at least corroborated.

[109] Expressed in terms of the arithmetico-symbolic representation aligned with the translation, the former interpretation makes the variable z the side of the small square, one seventh of the side of the first original square. According to the latter interpretation, z is the ratio between the sides of the original and the auxiliary squares.

Another interesting point is the use of the singular genitive in line 2. True enough, H. W. F. Saggs suggests that we have to do with a simple writing error, but that appears to be excluded by the singular -bi in line 3.[110] The small squares appear to be regarded as repetitions of an identical entity—a unit of accounting. Even in this respect, the present and the previous text are related.

V.8. VAT 8390, Nº 1 (MKT I, 335f.; cf. TMB, 112f.)

A final homogeneous second-degree problem is VAT 8390, Nº 1[111]:

Obverse I

$xy=10'$	1. Length *and* width *I have made span:* 10' the surface [uš *ù* sag] *uš-ta-ki-il-ma* 10 a-šà
$x^2=9\cdot(x-y)^2$	2. The length *to itself I have made span:* [uš *a*]*-na ra-ma-ni-šu uš-ta-ki-il-ma*
	3. A surface *I have built* [a-šà] *ab-ni*
	4. *So much as* the length over the width *goes beyond* [*ma*]*-la* uš u-gù sag *i-te-ru*
	5. *I have made span, to* 9 *I have repeated:* *uš-ta-ki-il a-na* 9 *e-ṣi-im-ma*
	6. *As much as that* surface *which* the length *by itself* *ki-ma* a-šà-*ma ša* uš *i-na ra-ma-ni-šu*
	7. *has been made span*[a]. *uš-t*[*a*]*-ki-lu*
	8. The length *and* the width what? uš *ù* sag en-nam
	9. 10' the surface pose 10' a-šà gar-ra
	10. *and* 9 (to) *which he*[b] *has repeated* pose: *ù* 9 *ša i-ṣi-pu* gar-ra-*ma*
$\sqrt{9}=3$ $[x=3\cdot(x-y)]$	11. The equilateral of 9 (to) *which he has repeated* what? 3. ib-si$_8$ 9 *ša i-ṣi-pu* en-nam
$x=3z$	12. 3 *to* the length pose. 3 *a-na* uš gar-ra

[110] See Saggs 1960: 139. According to SLa § 101, the use of -bi as a plural possessive suffix is apparently restricted to collective nouns ("people" and the like), and the same holds for the use of the singular status rectus after numbers above 10 (GAG § 139h). Strictly speaking, then, we have to do with either 16 copies of the same square or 16 practically identical squares.

[111] Nº 2 of the same tablet is a strict parallel—translated into symbolic algebra, the condition $x^2=9\cdot(x-y)^2$ is replaced by $y^2=4\cdot(x-y)^2$. The parallelism makes all restitutions of damaged passages certain.

I follow the improved readings given by Thureau-Dangin (1936: 58, repeated in TMB).

$\bar{y}=[x=]\,3z$	13. 3 *to* the width pose. 3 *a-n*[*a* s]ag gar-ra
$[(x-y)={}^1\!/_3 x=1\cdot z]$	14. *Since "so much as* the length over the width *goes beyond* *aš-sum ma-*[*la* uš] u-gù sag *i-te-ru*
	15. *I have made span", he has said* *uš-ta-k*[*i-il*] *iq-bu-ù*
$y=\bar{y}-(x-y)=3z-z$	16. 1 *from* 3 *which to* the width *you have posed* 1 *i-na* [3 *ša-a-n*]*a* sag *ta-aš-ku-nu*
	17. *tear out:* 2 *you leave.* *ú-*[*sú-uḫ-m*]*a* 2 *te-zi-ib*
$y=2z$	18. 2 *which you have left to* the width pose. 2 *ša t*[*e-z*]*i-bu a-na* sag gar-ra
$xy=3z\cdot 2z=6z^2$	19. 3 *which to* the length *you have posed* 3 *ša a-na* uš *ta-aš-ku-nu*
	20. *to* 2 *which to* the width *you have posed* raise, 6 *a-na* 2 *ša* ⟨*a-na*⟩ sag *ta-aš-ku-nu* íl 6
$6^{-1}=10'$	21. The igi of 6 *detach:* 10′. igi 6 *pu-ṭur-ma* 10
$z^2=10'\cdot 10'=1'\,40^\circ$	22. 10′ *to* 10′ the surface raise, 1′ 40°. 10 *a-na* 10 *a*-šà íl 1, 40
$z=\sqrt{1'\,40^\circ}=10$	23. The equilateral of 1′ 40° what? 10. íb-si$_8$ 1, 40 en-nam 10

Obverse II

$x=3z=3\cdot 10=30$	1. 10 *to* 3 *which to* the length *you have posed* 10 *a-na* 3 *š*[*a a-na* uš *ta-aš-ku-nu*]
	2. raise, 30 the length. íl 30 u[š]
$y=2z=2\cdot 10=20$	3. 10 *to* 2 *which to* the width *you have posed* 10 *a-na* 2 *ša a-na* sag *ta-aš-*[*ku-nu*]
	4. raise, 20 the width íl 20 sag
Proof:	5. *If* 30 the length, 20 the width *šum-ma* 30 uš 20 sag
	6. the surface what? a-šà en-nam
$xy=30\cdot 20=10'$	7. 30 the length *to* 20 the width raise, 10′ the surface. 30 uš *a-na* 20 sag íl 10 a-šà
$x^2=30\cdot 30=15'$	8. 30 the length *together with* 30 *make span:* 15′ 30 uš *it-ti* 30 *šu-ta-ki-il-ma*
$x-y=30-20=10$	9. 30 the length over 20 the width *what goes beyond*? 10 *it goes beyond.* 30 uš u-gù 20 sag *mi-nam i-tir* 10 *i-tir*

$(x-y)^2=10\cdot 10=1'\,40^\circ$	10. 10 *together with* 10 *make span:* 1′ 40°. 10 *it-ti* [10 *šu*]*-ta-ki-il-ma* 1, 40
$9\cdot(y-x)^2=9\cdot 1'\,40^\circ=15'$	11. 1′ 40° *to* 9 *repeat:* 15′ the surface. 1, 40 *a-na* 9 *e-ṣi-im-ma* 15 a-šà
$x^2=9\cdot(x-y)^2$	12. 15′ the surface is *as much as* 15′ the surface *which* the length 15 a-šà *ki-ma* 15 a-šà *ša* uš
	13. *by itself has been made span.* *i-na ra-ma-ni-šu uš-ta-ki-lu*

[a] Taken by itself, the phrase "*ša* uš *ina ramanišu uštākilu*" could perhaps also be interpreted as "which I made the length span by itself". The preposition *ina* occurs, however, in connection with *šutākulum* in all four occurrences of the relative clause in question and nowhere else in the tablet (nor anywhere else, as far as I can find out). Elsewhere in the tablet *šutākulum* stands with *u*, *ana* and *itti*. The propability that this distribution should have come about randomly is extremely small ($2.3\cdot 10^{-4}$ in a reasonable stochastic model). Furthermore, the occurrences in obv. II, 12f. and rev. 23f. stand in passages where the context requires the second person singular (because imperatives are pointed at) if the subject of the clause is not uš. Hence, the form cannot be the usual Št (II) (causative, reflexive), but must be Št (I) (passive of causative), of which this preterite form coincides with that of Št (II).
[b] The choice of "he" instead of "9" as the subject of the doubling is enforced by related passages in VAT 8520, obv. 7, 9, 11, rev. 8, 10.

As usually, the main lines of the procedure can be mapped by the arithmetical representation. On a number of points, however, it is inadequate: Why is a width equal to the length of 3 introduced in I, 13 (if this is at all the meaning of the expression "pose to"?)? Which principles govern the use of the three multiplicatory terms ("making span"; "raising"; and "repeating to n"? Why are so many different entities spoken of as "surfaces"? Normally, such words stand as epithets which serve to identify a number; this is also the case in I, 22, where "10′ the surface" is kept apart from "10′ [the igi of 6]". But this function can only be hindered when x^2 and $9\cdot(x-y)^2$ are also labeled "surface" (I, 2f.; II, 11f.).[112] So, in some sense or other, all these entities must be "surfaces".

Further: Why are the "surfaces" "built", while other complex expressions are not?[113] And why are "posing" (e.g. "posing 10′ the surface", in I, 9) und "posing

[112] In AO 8862 № 1 (translated below, section VIII.2), even the inhomogeneous expression $xy+x-y$ is a "surface"; so, the meaning of the term cannot be that of "product". Linear expressions, on the other hand, are never called surfaces; so, a generalized sense of "function" or "combined expression" is equally excluded. The sense "polynomium of the second degree" would of course be adequate, but much too abstract to be expected in a Babylonian context.

[113] Indeed, with one exception, only "surfaces" are "built" in Old Babylonian algebraic texts (VAT 8390 and AO 8862 in MKT I; YBC 4608 in MCT; TMS XVII). The exception concerns IM 52301, the deviations of which from normal usage were already mentioned above (note 95) (cf. also below, section X.1.).

to" (e.g. "posing 3 to the length", in I, 12) carefully distinguished all the way through the tablet? All these finer points of the formulation make no sense in the arithmetical interpretation. Several appear to call for a geometric reading, and indeed, a geometric representation answers all the questions, while at the same time giving us supplementary insight in the relation between "raising" and "making span".

The geometric representation which appears to be described in the text is shown in Fig. 10, the relation of which to the 16 squares of BM 15285 N° 10 is

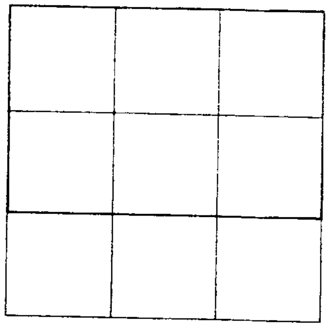


Figure 10. The geometrical interpretation of VAT 8390 N° 1.

obvious. The "repetition to 9" of the square on the excess of length over width is clearly seen to be a concrete repetition, no multiplicatory calculation. A width related to the number 3, and another width similarly related to 2, are clearly seen on the figure. And of course, all the "surfaces" are indeed surfaces in the most literal sense.

We observe that the numbers which are "posed" in I.9–10 are "real values"–the real surface of the rectangle, and the number of repetitions of the small square. The numbers which are "posed to" length and width (in I.12, 13 and 18), on the other hand, are not real values of the lengths and widths in question. It might seem as if "false values" (in the sense of a "false position") were "posed to" the entity for which they are assumed; still, according to normal Babylonian usage, later references (like that of I.19) could then be expected to quote the assumed numbers as values ("3 the length which you have posed", or perhaps "3 the false length which you have posed"). So, we are led towards the interpretation that "posing x to A" means "writing the number x along the entity A", as it was suggested in Fig. 9 (cf. note 108). Once again, the interpretation of the procedure of BM 13901 N° 10 as a subdivision rather than a comparison with an auxiliary figure is supported.

In one respect, the geometric interpretation changes the expectations which might be derived from the previous examples. When length and width, length together with length or excess together with excess give rise to rectangles or squares in I.1–5 they are "made span". So also in the proof, in II.8, 19, when the length and the excess are squared. But in I.20, the number of small squares is calculated by "raising 3 to 2", and in II.7, "30 the length" is "raised to 20 the width". What is the difference? Are the terms synonymous in spite of all contrary evidence?

The clue has to do with the term "to build", and with the way triangular and trapezoidal areas are found. Only when a length and a width (or two other lines) have been "made span", is a surface said to have been "built". Conversely, when the area of a triangle, a trapezium or a trapezoid is calculated, the term used is invariably "raising". So, firstly, the terms cannot be synonymous. And, secondly, one of them must belong with the process of building and the other with calculation. In other word, the process "to make span" is to be understood literally, as a process of *construction*, and to "build" means "to construct" (in agreement with the Latin etymology of the latter word). "Raising", on the other hand, means "calculating by multiplication".

This agrees well with the use of the terms in our text. In the beginning, the rectangle, the square on the length and the square on the excess are all constructed anew–none of them existed before. The number following the construction measures the area of the surface constructed–so, the calculation of this area is implied by the construction process[114], but it remains something different. In I.20, when the numbers 2 and 3 are multiplied and the number of small squares in the rectangle thus calculated, the rectangle is already there; hence, 3 is "raised to" 2, they are not "made span". Cf. also BM 13901, No 23: the *wāṣitum*-corner is already there, there is no need to construct it, nor is the *wāṣitum* "made span" (see above, note 101).

In the proof, the rectangle is still supposed to be there. In II.7, the length is "raised to" the width. The squares on length and excess, on the other hand, are "spanned". Since the same pattern repeats itself accurately in the second problem, this can hardly be an accident. So, the squares are not there to the same extent as the rectangle–either because only the rectangle is drawn, while the other figures are only imagined (3 and 2 being "posed" successively to the same width?), –or because everything is imagined, but the rectangle is more familiar as the basic figure and therefore still present to the inner eye. In any case it is made plausible that no complete figure like that of Fig. 10 was really drawn. Part of the procedure, if not all of it, was performed as mental geometry.

VI. The question of drawings

At this point it seems natural to ask whether the Babylonians left any traces of drawings like those of Fig. 4 to 10. The answer is, if we confine ourselves to algebraic texts like those to which these figures belonged[115], a clear no.

This might seem to present a problem to the geometrical hypothesis. Truly much "geometric" manipulation can have been performed mentally (and part of it must have been so performed, it appears from the above). But skill in mental geometry can only be acquired through familiarity with materialized geometry. So, a geometric interpretation of Babylonian algebra implies as its basis a physically palpable representation of this geometry.

[114] In AO 8862 (see section VIII.2), the calculation is at times made explicit as a separate process after the construction.

[115] The case of BM 15285 № 10 (see above, section V.7) is different. The whole tablet deals with areas of indubitably geometrical figures; no scaling and no cut-and-paste procedures appear to be involved.

On the other hand, drawings are also absent from the tablets in other cases where we can be sure that the argument presupposes a geometric figure. True enough, some real geometric problems are accompanied by a drawing. Still, this drawing is only an illustration of the statement of the problem, not of the procedure. Even in cases where auxiliary lines or appended figures are supposed by the argument they are left out from the drawing.[116] Furthermore, when the verbal statement of a geometric problem appears to be sufficiently clear, the sketch of the geometric situation is often dispensed with.

Even in cases where we can be sure that drawings have been made, they are thus absent from the tablets. This raises the question, where else they can have been made? Which medium can be imagined where drawings would leave no archaeological traces?

Several possibilities are open. The Greek drawings made in the sand are, at least from the anecdotes concerning the death of Archimedes, part of general lore.[117] For Mesopotamia, too, the use of the sand of the school courtyard has been proposed, namely as the medium for models of cuneiform signs in the basic scribal education.[118] Still, another possibility suggested by the Greeks is perhaps more interesting: The dust abacus, or its cognate, the wax tablet. As explained above (Chapter II), the Greek term ἄβαξ, "abacus", is in all probability derived from the semitic root *'bq*, "to fly away", "light dust". On that background it seems plausible that the Greeks have first met the abacus in the form of a dust-board, and that they have done so in the Western Semitic area.[119] As cultural connections between Syria and Mesopotamia were numerous–even much of the metrological system was shared and eventually taken over by the Greeks–use of the same device in Mesopotamia is at least a strong possibility. As to the wax tablet, it was certainly used in Mesopotamia in later times.

Whatever the medium of drawings corresponding to the solution of definitely geometric problems may have been, it left no traces, at least no traces which have been discovered until now. So, we need not worry much because no drawings corresponding to the solution of algebraic problems have been excavated.

[116] So in VAT 8512 (MKT I 341, cf. Gandz 1948: 36f. or Vogel 1959: 72), an auxiliary rectangle is attached to the triangle spoken of in the enunciation. In this text, by the way, even the verbal explanation which states the problem is left without the support of a sketch of the situation. Indeed, the problem as stated is clear and unambiguous and requires no sketch. The far less clear exposition of the procedure (less clear at least to modern interpreters) has not given rise to any explanatory drawing.

[117] Admittedly, the association of Archimedes with drawings in the sand are probably due to an ancient misunderstanding (see Dijksterhuis 1956: 30–32). Still, this very misunderstanding shows that geometrical drawings were at times made in the sand. The same is clear from an anecdote told by Vitruvius (*De architectura* VI, i, the story of the shipwrecked philosopher Aristippus finding geometric figures in the sand of the Rhodian shore).

[118] In the Old Babylonian school excavated in Tell ed-Dēr, the exercise tablets of the higher teaching levels contain the instructor's model and the student's attempt to imitate in parallel. The tablets belonging to the elementary level (stylus exercises, "Silbenalphabet A", "Syllabar a") contain no instructor's model, and Tanret (1982: 49) proposes that the models have instead been drawn "dans le sable de la cour".

[119] Even though Proclos is not very reliable as a source for the early period in Greek mathematics, his statement could be mentioned that arithmetic was first developed by the Phoenicians (*In Primum Euclidis . . . Commentarii* 65^{3-5}).

On the other hand, drawings have been excavated which show us something about the probable character of the geometric support for algebraic as well as geometric problem solution,–to wit the field plans. The autography of one of these, as well as a redrawing in correct proportions[120], would show us how.

The first feature of the plan to be observed is perhaps the subdivision into right triangles, right trapeziums and rectangles. Subdivisions are of course not easy to do without when a natural area has to be measured, but the plan shows

– that right triangles and trapeziums were looked for, not any triangle and trapezium. In the latter case, a height would have to be measured; right figures, on the other hand, are fully described by length and width (in the case of right trapeziums two widths, "upper" and "lower").

– that the right angles of the partial figures were clearly marked on the figure, while no care was taken to render other angles correctly.[121]

– and that the Babylonians were perfectly aware of the possibility to use auxiliary lines which were calculated, not measured. The calculation also shows awareness of the imprecision arising during measurement, since the dimensions of the partial figures are calculated in two different ways and the average found–whence the two writing directions for the partial areas.

Another striking feature is the total lack of care for a faithful rendering of proportions. A line is, so it seems, described by the number written unto it, if it is a line of importance for the determination of "lengths" and "widths" of the partial figures. One and the same line on the figure can even have two different numbers written unto it–this is the case of the line delimiting the two triangles to the uttermost left: the numbers alone tell us that two different lines in the terrain are meant.

This lack of care for correct proportions has some curious effects. At bottom of the plan, the hypotenuse of a right triangle continues directly as the skew side of a trapezium. In reality, the two lines are at an angle somewhat below 120°. To state things a bit sharply, the Babylonians did not make a drawing of the terrain in their field plans: They made a structural diagram, showing relevant lines, stating their lengths by inscribed numbers, and indicating their mutual relation with respect to the intended area calculation by visually right angles between lengths and widths.

Similar structural diagrams are also often made as a support for the verbal statement of geometrical problem texts. A glaring example of the difference

[120] A plan of the fields belonging to the district Šulgi-sipa-kalama, from the tablet MIO 1107, published, redrawn and discussed by Thureau-Dangin (1897).

[121] So, the repeated claims of S. Gandz (e.g. 1939: 415ff.), F. Thureau-Dangin (e.g. TMB xvii) and E. M. Bruins (e. g. TMS 4) that the Babylonians possessed no concept corresponding to our concept of quantifiable angles is not contradicted by the field plan. In all probability, the claims are correct for the Old Babylonian period. So, a theoretical concept of the right angle must also be considered absent. But clearly, a practical concept of the right angle, as the correct angle relevant for area measurement, must have existed according to the field plan and according to much other evidence, including architectural structures and the expression "the four winds", i.e. four cardinal points. Somewhat pointedly, a Babylonian "right" angle can be claimed to be the opposite of a "wrong" angle.

20*

between the real figure and the diagram interpreting the structure of the problem is YBC 4675.[122]

Naturally, this does not mean that the Babylonians were unable to make real geometrical drawings when they wanted to, or that they did not recognize a geometrical square. This is shown by the rich variety of geometrical forms drawn on the table BM 15285, of which one example was discussed in section V. 7. Still, the use of structural diagrams instead of drawings in the field plans and in the geometrical problem text suggests that the geometrical drawings or imaginations which possibly supported the solution of algebraic problems may very well have been of the diagram type. The first step in the reduction of Fig. 6, the redrawing in reduced vertical scale, need not have been performed in drawing. At the evidence of field plans etc. we may surmise that the Babylonians can have been able to imagine the left section of the unshaded part of the figure first as a rectangle and next as square, while the right section would in both steps be considered an appended "one third of the side". At the same time, they will have known that the changed conception of the whole figure would correspond to a reduced area: No longer 20′ but 40′ · 20′ = 13′ 20″.

Before leaving the problem of "drawings" we should take note of the fact that geometrical configurations can be represented materially by other means than through lines traced on a soft or colour-receiving surface. Some details of the Babylonian formulations could be read as hinting at a representation through small sticks or pieces of reed. I think especially of the identification of rectangular figures and their side and of the bisection through "breaking". It is also possible to make a pebble-representation of geometric figures in Greek style and to perform naive-geometric "algebraic" argumentation on such figures–and there exists indeed some vague evidence that early Greek calculators did so, inspiring thereby the development of the theory of figurate numbers.[123]

So, even though lines traced in sand, dust or wax appear to be the most plausible candidates for a representation of naive-geometric algebra it should be remembered that they are not the only possible candidates.

VII. The first degree

All texts discussed up to this point were of the "second degree", if we translate them into modern formalism, and such problems are the main concern of the whole investigation. To a large extent, however, Babylonian mathematics dealt with real-life problems, which in the Babylonian context were of the first degree; furthermore, the more complex second-degree-problems involve transformations and equations of the first degree. Both in order to locate the use of naive-geometric

[122] MCT 44f. and Plate 26.

[123] I deal with this question in my 1988: 24ff. It should be observed, firstly, that the multi-digit numbers occurring in many Old Babylonian algebra problems make them unsuited for precise representation through pebble patterns; and secondly, that the Babylonian procedure descriptions do not fit the most natural progress of a solution by pebbles.

methods correctly in relation to the complete structure of Babylonian mathematics and in order to grasp the methods of the complex second-degree-problems it is therefore of importance to get some idea of the techniques and ways of thought of Babylonian first-degree mathematics.

The present chapter presents two groups of texts suited for that purpose. Firstly I present two procedure-texts stemming from a larger group of problems all built on the same concrete data; they are sufficiently complex to admit of some insight into the patterns of thought employed. Secondly come two texts (stemming from a single tablet) reporting a didactical explanation of the transformations of a first-degree-"equation".

On the basis of the insights gained from these texts it will be possible to proceed to further second-degree-problems involving supplementary first-degree-transformations, which will give us a more complete picture of the relations between first- and second-degree-techniques.

VII.1. VAT 8389 N° 1 (MKT I, 317f.; improvements from Thureau-Dangin 1936: 58)

The problem deals with a domain composed of two partial fields of areas S_i and S_{ii}. The first field yields a rent in kind amounting to $r_i = 4$ gur of grain per bur, while the second yields $r_{ii} = 3$ gur per bur.[124] In the present problem, the total area is told to be $S_i + S_{ii} = 30'$ (sar), while the difference between the total rents yielded by the two fields is given as $R_i - R_{ii} = 8'\ 20^\circ$ (sila). (1 bur = 30' sar, 1 gur = 5' sila).

Obverse I

$r_i = 4$ gur/bur	1. *From* 1 bur 4 gur of grain *I have collected.* *i-na* bùr[gán] 4 še-gur *am-ku-us*
$r_{ii} = 3$ gur/bur	2. *From* 1 *second* bur 3 gur of grain *I have collected.* *i-na* bùr[gán] *ša-ni*[*-im*] 3 še-gur *am-ku-us*
$R_i - R_{ii} = 8'\ 20^\circ$ (sila)	3. The *grain* over the *grain* 8° 20′ *goes beyond.* *še-um* u-gù *še-im* 8, 20 *i-tir*
$S_i + S_{ii} = 30'$ (sar)	4. *My* meadows[a] I have accumulated: 30'. garim-*ia* gar-gar-*ma* 30
	5. *My* meadows what? garim-*ú-a* en-nam
The value of the practical unit bur is "posed" repeatedly in the	6. 30' the *bur* pose. 20' the *grain which he has collected* pose. 30 *bu-ra-am* gar-ra 20 *še-am ša im-ku-sú* gar-ra

[124] The verb translated "to collect" in my translation is *makāsum*, "Ertragungsteil, -abgabe einheben". MKT reads *malāsum*, "ausrupfen"; I follow Thureau-Dangin's correction (1936: 58), which shows the perspective to be not that of the peasant or the overseer-scribe but that of the landlord or his accountant. Neither this nor F. Thureau-Dangin's other corrections interferes with the mathematical structure of the text (cf. also MKT III 58).

"mathematical" unit sar, while the specific rents are posed directly, without the intermediate calculation, as
$r_{i}[=4\cdot 5']=20'$ (sila/bur)
$r_{ii}[=3\cdot 5']=15'$ (sila/bur)
Similarly, $R_i - R_{ii}$ and $S_i + S_{ii}$ are "posed"

The total surface $S_i + S_{ii} = 30'$ (sar) is bisected into two partial fields of 15' and 15', and the respective rents are calculated under the assumption that the original specific rents hold good for these two fields:

First the specific rents are recalculated in units of sila/sar (expressed as "false grain").

Next the hypothetical total rents R'_i and R'_{ii} are found through multiplication with the hypothetical areas of 15' (sar):

$R'_i = 10'$ (sila)

$R'_{ii} = 7'\ 30°$ (sila)

The difference between the hypothetical total rents is found:
$R'_i - R'_{ii} = 10' - 7'\ 30° = 2'\ 30°$

7. 30' the *second bur* pose.
   30 *bu-ra-am ša-ni-am* gar-ra
8. 15' the *grain which he has collected* pose.
   1[5] *š[e-a]m š[a] im-ku-sú* gar-ra
9. 8' 20° *which* the *grain* over the *grain goes beyond* pose
   8, 20 *š[a] še-um* u-gù *še-im i-te-ru* gar-ra
10. *and* 30' the *accumulation of* the surfaces of the meadows[b] pose
    ù 30 *ku-mur-ri* a-šà garim-meš gar-ra-*ma*
11. 30' the *accumulation of* the surfaces of the meadows
    30 *ku-mur-ri* a-šà garim-meš
12. *to two break:* 15'.
    a-na ši-na ḫi-pí-ma 15
13. 15' *and* 15' *until twice* pose:
    15 *ù* 15 *a-di ši-ni-šu* gar-ra-*ma*
14. The igi *of* 30', the *bur, detach:* 2″.
    igi 30 *bu-ri-im pu-ṭur-ma* 2
15. 2″ *to* 20', the grain *which he has collected*
    2 *a-na* 20 še *ša im-ku-sú*
16. raise, 40', the false *grain*; *to* 15' *which until twice*
    íl 40 *še-um* l[ul] *a-na* 15 *š[a] a-d[i] ši-ni-šu*
16a. *you have posed*
    ta-aš-ku-nu
17. raise, 10' *may your head retain.*
    íl 10 *re-eš-ka [l]i-ki-il*
18. The igi *of* 30', the *second bur, detach:* 2″.
    igi 30 *bu-ri-im ša-ni-im pu-ṭur-ma* 2
19. 2″ *to* 15', the *grain which he has collected*
    2 *a-na* 15 *še-im ša im-ku-sú*
20. raise, 30', the false *grain*: *to* 15' *which until twice*
    íl 30 *še-um* lul *a-na* 15 *ša a-di ši-ni-šu*
20a. *you have posed* raise, 7' 30°.
    ta-aš-ku-nu íl 7, 30
21. 10' *which your head retains*
    10 *ša re-eš-ka ú-ka-lu*
22. over 7' 30° *what goes beyond*? 2' 30° *it goes beyond.*
    u-gù 7, 30 *mi-nam i-tir* 2, 30 *i-tir*

This difference falls $8'\ 20° - 2'\ 30° = 5'\ 50°$ short of the real difference

23. 2' 30° *which it goes beyond from* 8' 20°
    2, 30 *ša i-te-ru i-na* 8, 20
24. *which* the *grain* over the *grain goes beyond*
    ša še-um u-gù *še-im i-te-ru*

Obverse II

The increase of the difference between the total rents is found for a transfer of 1 sar from the second to the first field: R'_i increases by 40', R'_{ii} decreases by 30', and hence the difference increases by $40' + 30' = 1°\ 10'$ (sila). The required total transfer is found through a division by 1° 10' to be 5' (sar), which is then added to the first hypothetical partial field and subtracted from the second in order to yield the real "meadows":
$S_i = 15' + 5' = 20'$ (sar)
$S_{ii} = 15' - 5' = 10'$ (sar)

Proof:
The total rents R_i and R_{ii} are found for the values $S_i = 20'$ sar, $S_{ii} = 10'$ sar (by renewed calculation of the "false grains")

1. *tear out:* 5' 50° *you leave.*
   ú-sú-uḫ-ma 5, 50 *te-zi-ib*
2. 5' 50° *which you have left*
   5, 50 *ša te-zi-bu*
3. *may your head retain*
   re-eš-ka li-ki-il
4. 40', the *ch[ange,] and* 30', [the *change*][d]
   40 *ta-ki-i[r-tam ù*[1] 30 [*ta-ki-ir*]*-tam*
5. accumulate: 1° 10'. The *igûm*[e] *I know not.*
   gar-gar-*ma* 1, 10 *i-gi-a[m ú-ul i-de]*
6. *What to* 1° 10' *shall I pose*
   mi-nam a-na 1, 10 *lu-uš-ku[-un]*
7. *which* 5' 50° *which your head retains gives me?*
   ša 5, 50 *ša re-eš-ka ú-ka-lu i-na-di-nam*
8. 5' pose. 5' *to* 1° 10' raise,
   5 gar-ra 5 *a-na* 1, 10 íl
9. 5' 50° *will it give you*
   5, 50 *it-ta-di-[k]um*
10. 5' *which you have posed from* 15' *which until twice*
    5 *ša t[a]-aš-ku-nu i-na* 15 *ša a-d[i] ši-ni-šu*
11. *you have posed, from one tear out*
    ta-aš-ku-nu i-na i[š]-te-en ú-sú-uḫ
12. *to* the *other append.*
    a-na iš-te-en ṣ[i]-im-ma
13. The *first* is 20', the *second* is 10'.
    iš-te-en 20 *ša-nu-um* 10
14. 20' is the surface of the *first* meadow, 10' is the surface of the *second* meadow.
    20 *a*-šà garim *iš-te-at* 10 a-šà garim *ša-ni-tim*
15. *If* 20' is the surface *of* the *first* meadow,
    šum-ma 20 a-šà garim *iš-te-at*
16. 10' the surface *of* the *second* meadow, *their grains* what?
    10 a-šà garim *ša-ni-tim še-ú-ši-na* en-nam
17. The igi *of* 30', the *bur, detach:* 2″.
    igi 30 *bu-ri-im pu-ṭur-ma* 2
18. 2″ *to* 20', the *grain which he has collected*
    2 *a-na* 20 *še-im ša im-ku-s[ú]*

19. raise, 40′; *to* 20′, the surface *of* the *first* meadow
íl 40 *a-na* 20 a-šà garim *i[š-te-at]*
20. raise, 13′ 20° the *grain, that of* 20′, the surface of the meadow.
íl 13, 20 *še-um ša* [20 a-šà garim]
21. The igi of 30′, the *second bur, detach:* 2″.
igi 30 *bu-ri-im ša-ni-[im pu-ṭur-m]a* 2
22. 2″ *to* 15′, the *grain which he has collected,* raise, 30′.
2 *a-na* 15 *še-[im ša im-ku-sú* í]l 30
23. 30′ *to* 10′, the surface *of* the *second* meadow,
30 *a-na* 10 a[-šà garim *ša-ni-tim*]
24. raise, 5′ the *grain, that of* 10′, the surface *of* the *second* meadow.
íl ⌊5⌋ *še-[u]m* [*ša* 10 a-šà garim *ša-ni-tim*]

Finally, the difference between the rents of the two meadows is found to be 8′ 20° as required.

25. 13′ 20° [(the grain of the first meadow)][f]
13, 20 [*še-um* (*ša*/a-šà) garim *iš-te-at*]
26. over 5 the *grain* [(of the second meadow)]
u-gù [5] *še*[*-im* (*ša*/a-šà) garim *ša-ni-tim*]
27. *what goes beyond?* 8′ 20° *it goes beyond.*
mi-nam i-tir [8, 20 *i-tir*]

[a] "Meadow" translates garim (∼*tawirtum*), "(Feld-)Flur, Umland, Umgebung". This name for a specific sort of field is possibly used because the normal name for a field (*eqlum*) is reserved in mathematical contexts for the meaning "surface" (cf. the last paragraph of section V.4). The same word is used for partial fields in VAT 8512 (see von Soden 1939: 148), in a context where parallel texts would make us expect A-ENGUR. This led Thureau-Dangin (1940a: 4f.) to the conjecture that the latter sign might in mathematical texts be a logogram for *tawirtum*, and not as usually (with the reading íd) for *nārum*, "Fluß, Wasserlauf, Kanal"; according to the Tell Ḫarmal compendium, however, the sign group was read *nārum*, "river" etc., even when a partial field was meant (IM 52916, rev. 15f., in Goetze 1951: 139).

[b] The plural of the "fields" is indicated by the suffix -meš, which in the living Sumerian language had been reserved to a plurality of persons (cf. Falkenstein 1959: 37). Obviously, the Sumerograms of the text are abbreviations for Akkadian words, and not evidence of an unbroken Sumerian mathematical tradition. Cf. also SLa, 63, § 76.

[c] "Grain" is in the nominative form, *še'um*. So, for once we are allowed by this happy apposition to interprete the common construction where a single number stands both as the result of one operation and as the object of the next: In the present case at least, the number is made explicit as a result, and is then implicitly understood in the next phrase.

This observation makes sense of a peculiar usage of the tablet BM 13901, viz. the use of the Sumerian agentive suffix -e as a separation sign between numbers. Indeed, O. Neugebauer made this explicit in his translation (e.g. in N° 1, Obv. I.1, translating the passage *ak-mur-ma* 45-e 1 *wa-ṣi-tam* as "habe ich addiert und 0;45 ist es. 1, den Koeffizienten". Since the suffix is only used when a separation of a result from a succeeding number is required, I chose to regard the main function of the sign as a separation indicator, and absorbed it into the interpunctuation of the translation. There is, however, little doubt that a secondary agentive connotation is also implied by the sign.

[d] "Change" translates *takkirtum*, my conjectural restitution of the damaged words of the line. Both O. Neugebauer and F. Thureau-Dangin suggest *ta-ki-il-tam*, because this word was known to them as a mathematical term, which seemed to make some sense, since they interpreted *šutākulum* simply as multiplication and *takīltum* hence as a "factor". The profounder understanding of the terms makes this reading meaningless and hence problematic. The only other word listed in AHw which seems to fit the remaining signs of the line is *takkirtum*, "Änderung". It is absent from other mathematical texts, but it turns out to make excellent sense in connection with a mathematical argument for which parallels are even more absent from our text material.

The term derives, indeed, from the D-stem of *nakārum*, viz. *nukkurum*, "(ver)-ändern", "bessern", "weitergeben", "anderswohin bringen"., etc. Now, in certain series texts the epithet kúr was applied to a "second" or "modified" width (cf. section IV.7). The Sumerogram is in general use for *nakārum* and its derivatives, but in the mathematical texts it appears to stand for the verbal adjective *nukkurum* of the D-stem. It is thus no wonder if the corresponding *nomen actionis* should belong to the mathematical idiom. Still, the restitution is conjectural. Truly, A. Westenholz finds it to fit the photograph at least as well as the old reading; but another trained eye, viz. that of W. von Soden, rejects it as impossible (personal communications).

[e] The text appears to distinguish the igi, i.e. the reciprocal of a number (an abstract mathematical concept), from the table value *igûm*, a very manifest entity. The latter term, in fact, turns up when the absence of the value from the table of reciprocals is stated. So does even the following text. Cf. YBC 6967, above, section V. 1, which deals precisely with table values.

[f] The double bracket [(. . .)] is used for a restitution of a passage where no parallel passages indicate the precise words of the original.

The mathematical commentary aligned with the translation shows that all steps of the procedure can be interpreted very concretely.[125] In principle, the text can of course also be followed by an abstract symbolic calculation, in the way its correctness is proved in MKT. But the text contains many steps which are superfluous if we suppose the real procedure to have been abstractly algebraic or arithmetical, so for instance the recalculation of the specific rents per sar in each case separately. The very complexity of the procedure points in the same direction: Why should the system

$$S_{\mathrm{i}}+S_{\mathrm{ii}}=30' \qquad \frac{5'}{30'}\,(4S_{\mathrm{i}}-3S_{\mathrm{ii}})=8'\ 20^{\circ}$$

[125] A concrete interpretation of the procedure was, as far as I know, first proposed by van der Waerden 1961: 67.

be solved via calculation of the quantity $S_{\mathrm{i}}-\frac{S_{\mathrm{i}}+S_{\mathrm{ii}}}{2}$? In the text discussed immediately below, a still more spectacular detour (as viewed from the standpoint of abstract algebra) will turn up. Finally, all problems from the group to which the present as well as the following text belongs can be followed in detail on the level of concrete thought. Even before we take the plausible use of the term *takkirtum* into account there seems to be little doubt that the real procedure is close to the one exhibited in the marginal commentary. If a collation confirms the possibility of the new reading, we can presumably regard the interpretation as fully confirmed, since no other replacement of the impossible *takīltum* seems at hand.

If we accept this conclusion, a number of features can be observed in the text. We observe that all intermediate quantities can be given a concrete meaning, either directly or, more significantly, with regard to a hypothetical situation. The "false grain" can be understood as "false" if we see it as that amount of grain which could be collected from the field in question had it been of area 1 sar; and the 2' 30° (sila) of obv. I, 22 can be interpreted as the difference in rents had the two fields been of equal magnitude.

The problem is of a type which in the Islamic Middle Ages might have been solved by a "double false position".[126] The present text avoids the technicalization inherent in this procedure and sticks to steps which can be intuitively and directly justified. The text keeps far from understanding via abstract arithmetical relationships; but it keeps equally far from the use of schemata learned by heart, and close to procedures which can be understood and explained.

Evidently, the problem is artifical. None the less, it appears to reflect the procedures of practical calculation very precisely. In order to see this we shall take note of some characteristics of Babylonian metrology. No metrological series were completely sexagesimal, and only weight measures approached sexagesimality. In order to make use of their tables of fixed constants and of the tables of multiples and reciprocals the scribes therefore had to convert the measures of practical life into sexagesimal multiples of a set of basic units (the nindan, the sar, the sila, etc.), which can be considered "mathematical" in the sense that they formed the basis for computation as performed by the scribes (but which were of course also practical units for measurements of a certain order of magnitude). In order to facilitate the conversions the scribes would make use of tables. This is precisely what happens in the present text. Areas and rents are given in the customary units bur and gur, which are of the relevant order of magnitude. In obv. 6 and 7 the scribe reads from his table that the bur is 30 (i.e., 30' sar), and that 4 and 3 gur are, respectively, 20 (i.e., 20' sila) and 15 (15' sila)–this is the reason that the last numbers can be stated directly. Double conversions, on the other hand (bur per gur into sila per sar) were not tabulated; therefore the specific rents (the "false grains") must be computed, as done in obv. 14–16 and 18–20,

[126] Then the difference in rent would have been calculated e.g. under the two different suppositions that $S_{\mathrm{i}}=S_{\mathrm{ii}}$ and $S_{\mathrm{ii}}=0$, and the real values of S_{i} and S_{ii} would have been derived by "inverse linear interpolation". Cf. Tropfke – Vogel 1980: 371f.
The difference between the procedure of the present problem and that of a "double false position" was already pointed out by Vogel 1960: 90ff.

and without preliminary conversion into "mathematical" units this could not be done by means of the table of reciprocals.

The closeness of the text to practical computation makes its treatment of the bur important. Both in the beginning, in obv. I, 17–21, and again in the proof, "the bur" and "the second bur" are distinguished. This implies that the value of the bur is not just taken note of as a number when it is "posed" in the beginning. It must be written down or represented in some other way in two different calculation schemes or concrete representation of the two fields.

We may compare this use of "posing" with that of obv. II, 6–9, the division of 5' 50° by 1° 10'. The double construction of line 8 shows that "posing" is different from the process of arithmetical multiplication, the "raising", but at the same time part of or presupposition for the performance of the computation–again, "posing" stands for the insertion into a computational scheme or other fixed procedure[127]–but not precisely the scheme in which the bur was posed.

A third function of the term is found in obv. I, 9f.: When $S_{\mathrm{i}}+S_{\mathrm{ii}}$ and $R_{\mathrm{i}}-R_{\mathrm{ii}}$ are posed, it can have nothing to do with fixed procedures–the entities $S_{\mathrm{i}}\pm S_{\mathrm{ii}}$ and $R_{\mathrm{i}}\sim R_{\mathrm{ii}}$ are dealt with differently in the set of related problems. Apparently, these fundamental entities are simply taken note of, presumably in writing, in any case by some material means. It is a fair guess that the way it is done is somehow analogous to the manner in which burs and reciprocals are "posed" in computational schemes or fixed representations.

Our guarantee that "posing" of a given quantity uses some material means is provided by obv. I, 17 and II, 3. In both places, intermediate results are to be "kept in mind", literally to be "held by the head". This is an expression which is only used for intermediate results, never when given quantities or quantities found by naive-geometric manipulations are taken note of. "Keeping-in-mind" appears to concern the recording of intermediate results which fall outside fixed procedures and computational schemes.

VII.2. VAT 8391 N° 3 (MKT I, 321f., improvements from Thureau-Dangin 1936: 58)

The two tablets VAT 8389 and VAT 8391 belong together, and contain a number of problems dealing with the same two fields. In the present problem, $S_{\mathrm{i}}-S_{\mathrm{ii}}$ und $R_{\mathrm{i}}+R_{\mathrm{ii}}$ are given, together with the values of the specific rents, which are common to all problems.

Reverse I

Given are again $r_{\mathrm{i}}=4$ gur/bur, and $r_{\mathrm{ii}}=3$ gur/bur	3. *If from* 1 bur of surface 4 gur of grain *I have collected*, *šum-ma i-na* bùr^gán a-[šà] 4 še-gur [*am-ku-us*]

[127] In those rather few cases which go "What shall I pose to Y which gives me X? Pose Z, X it gives you", the "raising" of Z to Y must then be considered as implied by the "posing" as an automatic consequence (cf. section IV.6.).

4. *from* 1 bur of surface 3 gur of grain *I have collected,*
i-na bùr[gán] a-šà 3 še-gur *am-[ku-us]*

Further $S_i - S_{ii} = 10'$ (sar)

5. *now* 2 meadows. Meadow over meadow 10' *goes beyond,*
i-na-an-na 2 garim garim u-gù garim 10 *i-tir*

$R_i + R_{ii} = 18'\ 20°$ (sila)

6. *their grain* I have accumulated: 18' 20°.
še-e-ši-na gar-gar-*ma* 18, 20
7. *My* meadows what?
garim-*ú-a* en-nam

The bur is "posed" (in sar) once for each meadow, and so are r_i and r_{ii} (in sila/bur)

8. 30' the *bur* pose. 20' the *grain which he has collected* pose.
30 *bu-ra-am* gar-ra 20 *še-am ša im-ku-sú* gar-ra
9. 30' the *second bur* pose. 15' the *grain which he has collected*
30 *bu-ra-am ša-ni-am* gar-ra 15 *še-am ša im-ku-sú*

9a. pose.
gar-ra

$S_i - S_{ii}$ is "posed" (the entity will be designated S' in the following)
$R_i + R_{ii}$ is "posed"

10. 10' *which* meadow over meadow *goes beyond* pose. 1[0 *š*]*a* garim u-gù garim *i-te-ru* gar-ra
11. 18' 20° the *accumulation of* the *grain* pose.
[18, 20 *ku-*]*mur-ri še-im* gar-ra

The *waṣûm* is "posed"

12. 1 the *wāṣûm*[a] pose.
[1 *wa-ṣi*]*-am* pose
13. the igi *of* 30', the *bur, detach:* 2″; *to* 20', the *grain which he has collected*
igi 3[0 *bu-ri-im pu-ṭur-m*]*a* 2 *a-na* 20 *še-im ša im-ku-sú*

The specific rent of the first meadow is recalculated in sila/sar

14. raise, 40', the false *grain; to* 10' *which* meadow over meadow *goes beyond*
íl 40 *še-um* l[ul *a-na* 1]0 *š*[*a* garim] u[-gù garim *i-te-r*]*u*

The rent R' of that part S' of the first meadow which exceeds the second meadow is found to be $R' = 6'\ 40°$. The remainder R'' of the total rent, $R'' = R_i + R_{ii} - R' = 11'\ 40°$, must then come from equal areas of the two meadows. Hence, a unit area is regarded; it is seen as composed of

15. raise, 6' 40°; *from* 18' 20°, the *accumulation of* the *grain*
íl 6, 40 *i-na* 18, 20 *ku-mur-ri še-im*
16. *tear out:* 11' 40° *you leave.*
ú-sú-uḫ-ma 11, 40 *te-zi-ib*
17. 11' 40° *which you have left, may your head retain.*
11, 40 *ša te-zi-bu re-eš-ka li-ki-il*
18. 1 the *wāṣûm to two break:* 30'.
1 *wa-ṣi-am a-na ši-na ḫi-pí-ma* 30
19. 30' *and* 30' *until twice* pose:
30 *ù* 30 *a-di ši-ni-šu* gar-ra-ma

$^1/_2$ sar from each meadow. These parts are "posed"

The specific rents r_i and r_{ii} are recalculated (r_i a second time) in sila/sar. The rents of the two halves of the unit sar are found, the first to be 20′

20. The igi *of* 30', the *bur, detach:* 2″; *to* 20', the *grain which he has collected*
igi 30 *bu-ri-im pu-ṭur-ma* 2 *a-na* 20 *še-im ša im-ku-sú*
21. raise, 40'; *to* 30' *which until twice you have posed*
íl 40 *a-na* 30 *ša a-di ši-ni-šu ta-aš-ku-nu*
22. raise, 20'; *may your head retain.*
íl 20 *re-eš-ka li-ki-il*
23. The igi *of* 30', the *second bur, detach:* 2″.
igi 30 *bu-ri-im ša-ni-im pu-ṭur-ma* 2
24. 2″ *to* 15', the *grain which he has collected*
2 *a-na* 15 *še-im ša im-ku-sú*

the second to be 15′

25. raise, 30′; *to* the *second* 30′ *which you have posed* raise, 15′.
íl 30 *a-na* 30 *ša-ni-[i]m ša ta-aš-ku-nu* íl 15
26. 15′ *and* 20′ *which your head retains*
15 *ù* 20 *ša re-eš-ka ú-ka-lu*

Hence, the rent of the average unit sar is 35′. Since the total rent of the area $(S_i - S') + S_{ii}$ can be taken to come from such average sars $(S_i - S' = S_{ii})$, and since it is known to be $R'' = 11'\ 40°$, $(S_i - S') + S_{ii}$ can be found through division by 35′ to be 20′.

27. accumulate: 35′; the *igûm I know not.*
gar-gar-*ma* 35 *i-gi-am ú-ul i-di*
28. *What to* 35′ *shall I pose*
mi-nam a-na 35 *lu-uš-ku-un*
29. *which* 11' 40° *which your head retains gives me?*
ša 11, 40 *ša r*[*e-e*]*š-ka ú-ka-lu i-na-di-nam*
30. 20' pose, 20' *to* 35′ raise, 11' 40° *will it give you.*
20 gar-ra 20 *a-*[*na*] 35 íl 11, 40 *it-ta-di-kum*

By error, this area 20' is not bisected, which would give $S_i - S'$ and S_{ii}. Instead, it is confused with the area of the first meadow (which is indeed known in advance to be 20'). S_{ii} is then found through the subtraction of $10' = S_i - S_{ii}$

31. 20' *which you have posed is* the surface *of* the *first* meadow.
20 *ša ta-aš-ka-*[*nu* a-]šà garim *iš-te-at*
32. *From* 20' the surface of the meadow, 10' *which* meadow over meadow *goes beyond*
i-na 20 a-šà garim 1[0 *ša*] garim u-gù garim *i-t*[*e*]*-ru*
33. *tear out:* 10' the surface *you leave.*
ú-sú-uḫ-ma 10 [a-šà *te-*]*zi-ib*

Reverse II

1–9 [contains a proof of no specific interest]

[a] *wāṣûm* is closely related to the *wāṣītum* of BM 13901, Nos 1, 2, 3 and 23 (cf. above, chapter V).

The basic conclusions could be repeated here: Once more, all more complicated steps in the calculation are chosen such that their results can be given a concrete meaning (and as before, simple transformations like that of bur/gur to sila/bur are performed without commentary). This time, however, there is direct and undamaged textual evidence for the correctness of the concrete interpretation given in the marginal commentary.[128] Firstly, of course, the 35′ of rev. I,27 must necessarily be the rent of an average sar; secondly, the rent of 20′ which corresponds to the semi-sar belonging to the first field is calculated with reference to "the bur", while the 15′ corresponding to the second field is calculated with explicit reference (in rev. I,23) to "the second bur", which all the way through belongs with the second field. The 35′ is clearly not the rent of an abstract average sar but that of a sar composed half from one and half from the other field.

This confronts us with a terminological problem: It appears that the bisection of rev. I,18 does not affect an area but instead a width of 1. Indeed, the *wāṣûm* which is already posed in rev. I,12, and which is later bisected, is nothing but the masculine form of the *wāṣītum* known from BM 13901, the width of 1 which transforms a length into an area of equal magnitude.

Evidently, the term is supposed by our author to refer to a familiar quantity. Like the bur, it is "posed" (in rev. I,12) for use in the calculation without being mentioned before among the given quantities.

The most obvious assumption is that the term means the same thing here as in the quadratic equations. If it does, we are provided with a clear exposition of the conceptualization of the calculation. The unknown area $(S_{\mathrm{I}}-S')+S_{\mathrm{II}}=s$ must be thought of as a rectangle of length s and width *1*. Half of it, of length s and width $^1/_2$ belongs to the first field, and the other half, of equal length and width, belongs to the second field. The 35′ should not then be thought of strictly as the rent of 1 average sar, but as the rent of 1 unit length (1 nindan) of the rectangle; similarly, the division of rev. II, 28–30 does not give us directly the area s, but instead the length s of the rectangle, and thereby implicitly its area.

The idea may seem strange to us. But a related conceptualization appears to lie behind the area unit eše (1 $\text{eše}_2=10$‘ sar). It corresponds to a field of width "1 rope" (1 $\text{eše}_3=10$ nindan) and length 1‘ nindan; another unit, the "(area) nindan", has the same length but only the width 1 nindan.[129] Similar ideas are also found in Egyptian area metrology (1 "cubit of land" being a rectangle of width 1 cubit and length 100 cubit = 1 "reel of chord", while 1 "thousand of land" had the same length and a width of 1000 cubits[130]) and in Babylonian measures of volume (identifying units of area and volume by means of a standard height equal to 1 cubit). So, the whole idea may have been most concrete to a Babylonian scribe, and hence the identification of *wāṣûm* and *wāṣītum* can be considered reasonable.[131]

[128] An explanation of the procedure which as far as I know has been overlooked by all previous investigators of the text.

[129] See Powell 1972: 185 and passim.

[130] See Peet 1923: 24f.

[131] It can be observed that the length of the bur when applied to the width *waṣûm* = 1 nindan equals the largest Babylonian length measure, the danna (≈ 10.8 km), as it was pointed out independently of the present analysis by M. Powell at the Third Workshop on Concept Development in Mesopotamian Mathematics, Berlin, December 1985.

We remember that it is precisely the idea that a linear extension possesses a "standard width" of *1* nindan which permits us to see an area calculation as an operation of proportionality or scaling, and which thus gives conceptual unity to all applications of the term "raising" (cf. Fig. 3 and the discussion of the meaning of the term in section IV.3).

VII.3. TMS XVI, parts A and B (TMS, 92, cf. von Soden 1964)

The two preceding texts treated seemingly concrete (if surely not practical) problems of the first degree. The present texts are very different. They deal with the basic abstract length-width-representation, and they solve no problems[132]; instead, they present us with a didactical discussion of the meaning and the transformations of simple "equations of the first degree". They have been excavated in Susa (late Old Babylonian epoch), and they belong to a type not known from Babylonia itself. Maybe the need to fix didactical explanations in writing have to do with the fact that the texts represent a cultural import, no continuous autochthonous tradition; maybe the Susa excavators have simply had good luck where those working on (or looting!) Babylonian sites have not.

Although the two texts are mutually independent, they are so close to each other that both translations are best given together, before the commentary.

Part A

$(x=30,\ y=20)$ $x+y-{}^1/_4y=45$	1. The *4th of* the width *from* the length and width to tear out, 45. You, 45 [*4-at* sag *i-na*] uš ù sag zi 45 za-e 45
4 · (– " –) = 3‘	2. *to* 4 *raise*, 3‘ *you see*. 3‘, *what* is *that*? 4 *and* 1 pose. [*a-na* 4 *i-ší* 3 *ta*]-*mar* 3 *mi-nu šu-ma* 4 ù 1 gar
$x+y=50,\ {}^1/_4y=5$ $4\cdot 5=[4\cdot {}^1/_4y=]\ 1\cdot y$	3. 50 *and* 5, to tear out[a], pose. 5 *to* 4 *raise*, 1 width. 20 *to* 4 *raise* [50 ù] 5 zi ⌈gar⌉ 5 *a-na* 4 *i-ší* 1 sag 20 *a-na* 4 *i-ší*
4 · 20 = 1‘ 20° = 4 · y 4 · 30 = 2‘ = 4 · x	4. 1‘ 20° *you see*, 4 widths. 30 *to* 4 *raise*. 2‘ *you see*, 4 lengths. 20, 1 width to tear out, 1, 20 *ta-⟨mar⟩* 4 sag 30 *a-na* 4 *i-ší* 2 *ta-⟨mar⟩* 4 uš 20 1 sag zi
1′ 20° – 20 = 4 · y – 1 · y = 1‘ 2‘ + 1‘ = [4 ·]x + 3 · y = 3‘	5. *from* 1‘ 20°, 4 widths, tear out, 1‘ *you see*. 2‘, lengths, *and* 1‘, 3 widths, ACCUMULATE, 3‘ *you see*. *i-na* 1, 20 4 sag zi 1 *ta-mar* 2 uš ù 1 3 sag UL.GAR 3 *ta-mar*
$4^{-1}=15'$ ${}^1/_4$ · 2‘ = ${}^1/_4$ · ([4 ·]x) = 30 = 1 · x	6. The igi of 4 *detach*, 15′ *you see*. 15′ *to* 2‘, lengths, *raise*, 30 *you see*, 30 the length igi 4 *pu-[ṭú-ú]r* 15 *ta-mar* 15 *a-na* 2 uš *i-ší* 3[0] *ta-⟨mar⟩* 30 uš

[132] True enough, the mathematical commentary in TMS claims that they do, and even tries to make them do it, though with considerable violence to the texts.

$^1/_4 \cdot 1^` = 15 = [^3/_4 \cdot\ ]y$ $[1 \cdot x + ^3/_4 \cdot y =]\ 30 + 15$	7. 15′ *to* 1` *raise*, 15 the *contribution*[b] *of* the width. 30 *and* 15 *retain*[c] (?). 15 *a-na* 1 *i-ši* [1]5 *ma-na-at* sag 30 *ù* 15 *ki-il*
The coefficient to y is found by an argument of type "single false position" to be $(4-1)/4 = 3/4 = {}^1/_4 \cdot 3 = 15' \cdot 3 = 45'$	8. *Since* "the *4th of* the width *to tear out*", *it has been said to you*[d], *from* 4, 1 tear out, 3 *you see*. *aš-šum* 4-*at* sag *na-sà-ḫu qa-bu-ku i-na* 4 1 zi 3 *ta-mar* 9. The igi of 4 *detach*, 15′ *you see*. 15′ *to* 3 *raise*, 45′ *you see*, 45′ *as much as* (there is) of widths. igi 4 *pu*-⟨*ṭú-úr*⟩ 15 *ta-mar* 15 *a-na* 3 *i-ši* 45 *ta*-⟨*mar*⟩ 45 *ki-ma* [sag]
The coefficient to x is 1 (from line 6) The "width" y of the calculation is known to be 1 times the "true width" (of a figure?); hence $y = 1 \cdot 20 = 20$, and $45' \cdot y = 45' \cdot 20 = 15$, which when subtracted from $45 = 30 + 15$ leaves $30 = 1 \cdot x$	10. 1 *as much as* of lengths pose. 20 the true[e] width *take*. 20 *to* 1 *raise*, 20 *you see*. 1 *ki-ma* uš gar 20 gi-na sag *le-qé* 20 *a-na* 1 *i-ši* 20 *ta-mar* 11. 20 *to* 45′ *raize*, 15 *you see*. 15 *from* ^{30}15 *tear out*, 20 *a-na* 45 *i-ši* 15 *ta-mar* 15 *i-na* ^{30}15 [zi] 12. 30 *you see*, 30 the length. 30 *ta-mar* 30 uš

[a] TMS transcribes the beginning of this line as [50 ù] 5 ZI.A(!) ⟨GAR⟩ and interpretes ZI as a (phonetically motivated) writing error for SI, which would give the passage the meaning "50 *and* 5 which go beyond ⟨pose⟩". The supposed A is, however, damaged and clearly separated from the ZI. As far as I can see from the autography, the traces might as well represent the lacking GAR, which would give the reading [50 *ù*] 5 zi gar, "50 and 5, to tear out, pose". Not only is this in harmony with the actual text, it also has the clear advantage over the reading of TMS to be in agreement with the zi, "to tear out", of line 4, as well as with those of lines 1, 5 and 8. The latter of these, which is an explicit quotation of line 1, is written in syllabic Akkadian, excluding any error. It is also this quotation which shows that the zi is thought of as an infinitive, not as a finite form (cf. below, note d).

[b] "Contribution" translates *manātum*, an abstract noun derived from *manûm*, "to count". Etymologically, the meaning would be "the count"/"the counting". However, the term is found only here and in two other Susa texts (TMS XII and XXIV). In one of these, its use is unclear, in the other the term is isolated by a break. AHw suggests hypothetically an identification with Hebrew and Aramaic *m*[e]*nat*, which in HAHw (pp. 438[2]–439[1]) is exemplified by "Anteil der Priester und Leviten" and "d. Teil (Beitrag) des Königs". The ensuing "share/contribution of the widths" fits the present text excellently, and it is not contradicted by the other two occurrences.

[c] "Retain" is a conjecture (*ki*!-*il*!) due to von Soden (1964: 49). TMS has *ḫulum*, Assyrian for "way", interpreted as "method" by the editors.

[d] This quotation is very remarkable, since the ideographic zi is rendered syllabically by an indubitable infinitive, *na-sà-ḫu*.

TMS claims that an indubitable gi-na, "true", must be a writing error for *ki-ma*, "*as much as*". If this were the case, *kima* sag, "*as much as* of widths", would represent both the coefficient to the width (45′, in line 9) and the value of the width (20, in line 10)!

Part B

$(x = 30,\ y = 20)$ $(x-y) + {}^1/_4 y = 15$	13. The *4th of* the width *to* that *which* length over width *goes beyond* to append 4-*at* sag *a-na ša* uš ugu sag *i-te-ru* daḫ
$4 \cdot (-\text{''}-) = 1^`$	14. 15. *You*, 15 *to* 4 *raise*, 1 *you see*, *what* is *that?* 15 za-e 15 *a-na* 4 *i-ši* 1 *ta-mar mi-nu-šu-*[*ú*] 15. 4 *and* 1 pose. {. . .} 4 *ù* 1 gar {15 *a-na* 4 *i-ši* 1 *ta-mar mi-*[*nu-šu-ú*]}
$x - y = 10,\ {}^1/_4 y = 5$	16. 15 *scatter*[a]. 10 the going-beyond *and* 5 the appended pose. 20 the width 15 *sú-pi-iḫ* 10 dirig *ù* 5 daḫ gar 20 sag
$(x-y) + y = 10 + 20$ $= 30 = x$ $4 \cdot {}^1/_4 y = 4 \cdot 5 = 20 = y$	17. *to* the going-beyond append; 30 the length, 20 to tear out pose. 5 *to* 4 *raise*, *a-na* 10 dirig daḫ 30 uš 20 zi gar 5 *a-na* 4 *i-ši*
$4 \cdot y = 4 \cdot 20 = 1^`\,20^\circ$	18. 20 *you see*; 20, the width, *to* 4 *raise*, 1` 20° *you see*. 20 *ta-mar* 20 sag *a-na* 4 *i-ši* 1, 20 *t*[*a-mar*]
$4 \cdot x = 4 \cdot 30 = 2^`$	19. 30, the length, *to* 4 *raise*, 2` *you see*. 20, the width, 30 uš *a-na* 4 *i-ši* 2 *ta-mar* 20 sag
$4 \cdot y - y = 1^`\ [= 3 \cdot y]$	20. *from* 1` 20° tear out, 1` [. . .] 1` *you see* [(3 widths(?)); 1`] *i-na* 1, 20 zi 1 [. . .] 1 *ta-mar* [. . . 1]
$4 \cdot x - (4 \cdot y - y) =$ $2^` - 1^` = 1^`$	21. *from* 2`, lengths, tear out, 1` *you see*, *what is that?* [. . .] *i-na* 2 uš zi 1 *ta-mar mi-nu šu-ú* [. . .{1(?) *ta*} . . .]
The coefficient to y is found by an argument of type "single false position" to be $(4-1)/4 = 3/4 = {}^1/_4 \cdot 3 = 15' \cdot 3 = 45'$, the "negative" (i.e. subtractive) type of which is noted.	22. *From* 4, *of* the *fourth*, 1 tear out, 3 *you see*. The igi of 4 *detach*, 15′ *you see*. *i-na* 4 *ri-ba-ti* 1 zi 3 *ta-mar* igi 4 *pu*⟨*-ṭú-úr*⟩ 15 *ta-*[*mar* 23. 15′ *to* 3 *raise*, 45′ *you see*, *as much as* of widths pose. Pose to tear out[b]: [15′] *a-na* 3 *i-ši* 45′ *ta*⟨*-mar*⟩ *ki-ma* sag gar gar! zi-*ma*
The coefficent to x is 1, $1 \cdot x = 1 \cdot 30 = 30$	24. 1 *as much as* of lengths pose [. . .] 1 *take*, *to* 1 length 1 *ki-ma* u[š gar . . .] 1 *le-qé a-na* 1 uš
$45' \cdot y = 45' \cdot 20 = 15$	25. [*raise*, 30 *you see* (. . .)][c]. 20 the width, 20 *to* 45′, widths, *raise*, [*i-ši* 30 *ta-mar* (. . .)] 20 sag 20 *a-na* 45 sag *i-ši*

$15+45'\cdot y=15+15$ $=30=x$	26. 15 *you see.* 15 *to* 15 append, 30 *you see,* 30 the length. [15 *ta-mar* 15] *a-na* 15 daḫ 30 *ta*⟨*-mar*⟩ 30 uš

[a] "To *scatter*" translates *sapāḫum*, "auflösen, zerstreuen" (the reading is due to von Soden–private communication, cf. 1964: 49). In fact, 15 is "scattered", i.e. analyzed into its constituent components 10 $(=x-y)$ and $5(=1/_4y)$.

[b] TMS reads "4 zi-*ma*" and neglects the "4" in the translation, since this number gives no sense. Often GAR (=gar, "to pose") and 4 cannot be distinguished; so, we seem to be left with the choice between a formulation which makes no sense in its context, but which could have crept in by a copying error (the reading of TMS) and a reading which makes sense, and which possesses a parallel in line 17 (the present reading). However, close inspection of the autography shows an outspoken tendency to write GAR symmetrically, while 4 is normally written asymmetrically (as and , respectively). Only collation could decide whether the few exceptions are due to the scribe or the copying, and whether the difference reflects a different sequence of impression of the wedges. In any case, the problematic sign is as much a GAR as its left neighbour. So, the reading gar zi-*ma* appears to be established beyond reasonable doubt. Cf. also part A, line 3.

[c] TMS makes a different restitution, which presupposes that *laqûm*, "to take", is used synonymously with *našûm*, "to raise" as a term for multiplication. This presupposition is totally unsupported, and clearly contradicted by part A, line 10.

The present restitution is conjectural–only the "raise" required by the "to" seems secure. Possibly the restitution fills out the entire lacuna, possibly a few more signs can have found their place.

Both parts deal with a length of 30 and a width of 20, and this is supposed by the text to be known in advance[133], as are the sum of length and width, the excess of length over width, and the fourth of the width.

Part A leads off with an equation which in symbolic translation runs $x+y-1/_4y=$ $=45$ and asks for the meaning of the *3*ˋ which result when the right-hand-side is multiplied by *4*. It then looks at the single members of the left-hand-side, multiplying each with *4*, explaining $4\cdot 20=1ˋ\ 20°$ to be $4y$, $4\cdot 30=2ˋ$ to be $4x$, and $4\cdot$ ([subtractive] $5)=20$ to be a subtractive y (cf. below on this indication of sign). The result is $2ˋ+(1ˋ\ 20°-20°)=$ (the required) *3*ˋ.

Then, from line 6 onwards, the reverse operation is performed, but this time on the sum of $2ˋ=4x$ and $1ˋ=3y$. $1/_4\cdot 2ˋ=30$ is told to be simply x, while $1/_4\cdot 1ˋ=15$ is told to be the "contribution of y". In line 8f., the coefficient of y is calculated to be $1/_4\cdot(4-1)=45'$, and it is given the name "as much as" (*kīma*) (there is) of widths. In line 10, the coefficient of x is stated to be *1*. Finally, the product of y

[133] According to TMS, only the width is known. Had this been the case, the operations of line A.3 would have proceeded conversely: From a width of 20 to its fourth (5), whence from 45 to 50. In part B, similar disagreements between the text and E. M. Bruins's assumption that the length be unknown can be pointed out.

and its coefficient is calculated and subtracted from the *45* of the right hand side (written as it was already analyzed in lines 6f.), and the remainder is seen to be equal to the length, as required.

Part B runs along similar lines, the main difference being perhaps that this time the analysis of the right hand side appears to be made verbally explicit as a "scattering" in line 16. "Contributions" and "coefficients" recur–the former, it is true, without the explicit label *manātum*.

For the sake of clarity, the operations can be organized schematically, as it is shown on the following page.[134] We observe that there is a close analogy between the Babylonian text and our own treatment of the corresponding equation. Not only the coefficients and the contributions but also the multipliers *1* and *4* of the left margin are stated explicitly. It seems, however, that most of the operations are supposed to be followed mentally: in part A, only the multipliers and the numbers *50* and *5* of line *δ* are "posed", in a way which suggests written representation; all the rest is done rhetorically, or followed without notation on a graphic representation.

In the previous texts the concrete pattern of thought was noticed. A similar observation can be made here, both on the terminology used for contributions and coefficients and for the way the coefficients are calculated. In both parts, the coefficient of y is found by an argument of type "single false position" and not through the arithmetically simpler but more abstract calculation $1-1/_4=$ $=1°-15'=45'$. Similar patterns are found elsewhere in the material, e.g. in VAT 7532, rev. 6f. (MKT I, 295).

Even if concrete, the designation of the coefficient by a special expression can be considered a formalization of the "accounting technique" which was discussed above (section V.6). Another formalization of something which was done currently with or without formalization is the designation of certain numbers or entities as "subtractive", "to tear out" (in lines 3, 4, 17 and 23), written by the sumerogram zi. That we are really confronted with sort of sign is most clearly demonstrated by lines 4 to 5, where "*20*, *1* width", is firstly given the epithet "to tear out", and afterwards really torn out.

zi is not only used to indicate subtractiveness but also for the subtractive operation ("tearing-out") itself, e.g. in line 1, as it is indicated by the preposition "from" (*ina*). It is an old issue whether such occurrences should be Akkadianized in transliterations. F. Thureau-Dangin did so without hesitation, regarding the sumerograms as pure logograms which were read by the scribes as grammatical Akkadian and which should hence be read so by us. He was so confident about this that he did not indicate the sumerogram parenthetically, as it is done in e.g. TMS. O. Neugebauer, on the other hand, claimed that the ideograms functioned as mathematical operators, not as words belonging to current language (see e.g. MKT I, viii). Line 8 of part A shows that O. Neugebauer was at least partly right: The statement is quoted, but the ideographic writing zi is rendered in phonetic writing as an infinitive, *na-sà-ḫu* (the text is written without "mimation", the final *m* of nouns and nominal verbal forms which was gradually dropped).

[134] The symbolic schematization of part A was proposed to me by P. Damerow at the First Workshop on Concept Development in Mesopotamian Mathematics, Berlin 1983, where I first presented my interpretation of the text.

At least the term zi must, at least in the Susa school, have been regarded as an ideogram for an abstract mathematical operation, not as a logogram to be provided with correct grammatical pre- and suffixes when read.

Indications exist that the restrictions to zi and to the Susa school are superfluous. Indeed, if íb-si$_8$ were read *mitḫartum* (as claimed by F. Thureau-Dangin), how are we to understand changes in the ideographic expression following Sumerian homophonic patterns (íb to ib, si$_8$ to si)? How are we to explain the use in certain texts (among which IM 52301, see below, section X.1) of a term *basûm*, evidently an Akkadianized pronunciation of ba-si$_8$? What are we, finally, to do about the distinction between the Akkadianization *igûm*, the table value, and igi, the abstract reciprocal number? It appears that certain Sumerograms were (at least in certain text-types, among which the compactly written series texts must be reckoned) regarded as ideograms, that they were sometimes read in Sumerian and sometimes Akkadianized without proper inflection in person and tense.[135]

A final observation on the text concerns part A, line 10f. Both the formulation and the actual calculation are conspicuous. Why is the width spoken of as a "true width"? And why is 45′ widths calculated not as 20 raised to 45′ but in two steps, the true width being first raised to 1, and the result next raised to 45′?

The immanent analysis of the text provides us with no answer; below we shall see how at least a suggestion can be found in the texts BM 13901 N° 14 and TMS IX (sections VIII.1 and VIII.3, respectively)—a suggestion which appears to be confirmed in TMS XIX (cf. below, note 176).

Symbolic and graphic schematization of the operations

$$
\begin{array}{lll}
\alpha & & 1x + \underbrace{1y - {}^1\!/_4 y} = 45 \\
\beta & & 1x + 45'y = 45 \\
\gamma & 1 & \underbrace{30 + 20} - 5 = 45 \\
\delta & & 50 \quad \underbrace{- \; 5} = 45 \\
\delta' & & 30 + 15 = 45 \\
\hline
\varepsilon & & 4x + \underbrace{4y - 1y} = 3^{\backprime} \\
\zeta & 4 & 4x + 3y = 3^{\backprime} \\
\eta & & 2^{\backprime} + \underbrace{1^\circ 20' - 1} = 3^{\backprime} \\
\vartheta & & 2^{\backprime} + 1^{\backprime} = 3^{\backprime}
\end{array}
$$

Apparently, the 1 and 4 posed in line 2 of the text are the factors written to the left of the two groups of equations. The rest of part A discusses the relations between the lines α to ϑ.

It is seen that α represents the original equation of "lengths" and "widths", written symbolically, while ε is obtained from this original equation through multiplication

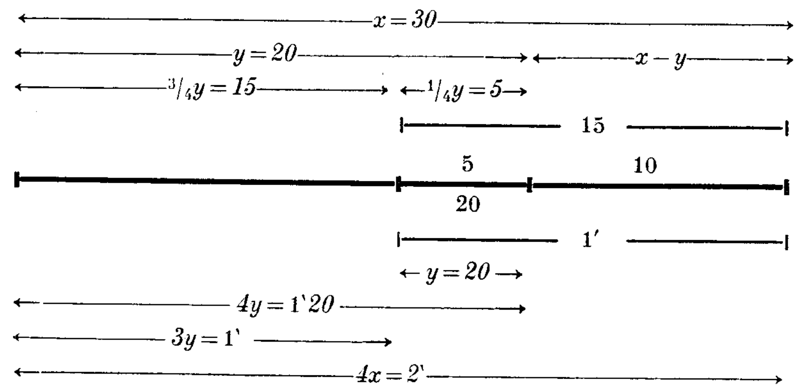
[135] On other occasions we are of course forced to acknowledge some Sumerograms as logograms for proper Akkadian, — viz. when they are provided with Akkadian phonetico-grammatic complements. Cf. note a to BM 13901, No 23 (section V.4).

by 4. γ and η represent the same equations when the known values of length and width are inserted.

In the text, line 3 "poses" the 50 and 5 of γ, representing 5 as "that which is torn out" (from 50). Next (line 3–5), the transformation of γ into η is explained term for term in order to solve the problem raised in line 2: which meaning to ascribe to the 3` which arise when the right-hand side of α is multiplied by 4. This is done with reference to ε, ζ and ϑ.

Line 6f. explains the reverse transformation η to γ, referring to δ′, where the respective contributions of lengths and widths are separated. Line 8–12, finally, explains δ′ in terms of β where the coefficients of x and y, i.e. "as much as there is" of lengths and widths, are found and multiplied by the numerical value of these entities.

Instead of this symbolic schematization, a graphic scheme could also be used. For the sake of variation we shall apply it to part B, which to a first glance seems somewhat more opaque than part A, but which turns out to be very simple in graphic representation:



Once again, the upper half of the scheme corresponds to the original equation and the lower half to the multiplication by four.

The steps of the text are easily demonstrated at the scheme. Evidently, an oral representation would not need the many lines drawn here. The heavy line in the middle could do, if only the teacher pointed out in each step which segment was spoken of now. While the symbolic scheme is of course anachronistic as a mapping of the text, the graphic representation may thus be close to what actually went on in the Susa school.

A graphic interpretation of part A will be found in my 1989: 24.

VIII. Combined second-degree problems

In chapter V, a number of simple second-degree problem texts were presented and discussed, and in chapter VII we had a look at some very concrete first-degree problems. Together, the two chapters might convey the impression that Babylonian mathematics was not only concrete in its cognitive orientation but also simple, not to say simplistic. In order to counteract at least in part this misleading impression the present chapter shall present a couple of texts which combine the first- and second-degree techniques in various ways, demonstrating a bit of the sophistication to which Babylonian algebra was able to rise while remaining concrete and "naive". The last section of the chapter presents another didactical Susa text, which builds the bridge from simple to more sophisticated second-degree algebra.

VIII.1. BM 13901, Nº 14 (MKT III, 3; cf. TMB, 5)

Several other problems from the same tablet were already presented above in Chapter V. The present problem contains yet another problem of squares, this time in two variables connected through a simple inhomogeneous equation of the first degree. Through substitution and use of the accounting technique, the problem is reduced to that dealt with in section V.5 and solved by the same procedure.

	Obverse II
$x^2+y^2=25`\,25^\circ$	44. The surfaces *of my two confrontations I have accumulated:* 25` 25°. a-šà *ši-ta mi-it-ḫa-ra-ti-ia ak-mur-ma* ⌜25,]25
$y={}^2/_3x+5$	45. The *confrontation, two-third of* the *confrontation and* 5 nindan *mi-it-ḫar-tum ši-ni-pa-at mi-it-ḫar-tim* [*ù* 5 ninda]n
$x=1\cdot z$ $y=40'\cdot z+5$	46. 1 *and* 40′ *and* 5 *overgoing* the 40′ *you inscribe.* 1 *ù* 40 *ù* 5 [*e-le-nu* 4]0 *ta-la-pa-at*
$x^2=1^2\cdot z^2=1\cdot z^2$	47. 5 *and* 5 *you make span,* 25 *inside of* 25` 25° *you tear out:*[a] 5 *ù* 5 [*tu-uš-ta-kal* 25 *lìb-bi* 25, 25 *ta-na-sà-aḫ-ma*]
	Reverse I
$y^2=(40'\cdot z+5)^2$ $=26'\,40''\cdot z^2$ $+2\cdot 40'\cdot 5\cdot z+25$	1. 25` *you inscribe.* 1 *and* 1 *you make span,* 1. 40′ *and* 40′ *you make span,* [25 *ta-la-pa-at* 1 *ù* 1 *tu-uš-ta-kal* 1 40 *ù* 40 *tu-uš-ta-kal*]
$1^\circ\,26'\,40''\cdot z^2$ $+2\cdot 5\cdot 40'\cdot z=25`$ Putting $Z=1^\circ\,26'\,40''\cdot z$ we get when multiplying by $1^\circ\,26'\,40''$	2. 26′ 40″ *to* 1 *you append:* 1° 26′ 40″ *to* 25` *you raise:* [26, 40 *a-na* 1 *tu-ṣa-ab-ma* 1, 26, 40 *a-na* 25 *ta-na-ši-ma*]
$Z^2+2\cdot 5\cdot 40'\cdot Z$ $=Z^2+2\cdot 3^\circ\,20'\cdot Z$	3. 36` 6° 40′ *you inscribe.* 5 *to* 40′ *you raise:* 3° 20′ [36, 6, 40 *ta-la-pa-at* 5 *a-na* 4]0 *t*[*a-na-ši-ma* 3, 20]
$=1^\circ\,26'\,40''\cdot 25`$ $=36`\,6^\circ\,40'$ $(Z+3^\circ\,20')^2=36`\,6^\circ\,40'$	4. *and* 3° 20′ *you make span,* 11° 6′ 40″; *to* 36` 6° 40″ *you append:* [*ù* 3, 20 *tu-uš-ta-kal* 11, 6, 40] *a-na* 3[6], 6, 40 [*tu-ṣa-ab-ma*]

$+(3^\circ\,30')^2$ $=36`\,17`\,46'\,40''$	5. 36` 17° 46′ 40″ makes 46° 40′ equilateral. 3° 20′ *which you have made span*
$Z+3^\circ\,20'=\sqrt{36`\,17^\circ\,46'\,40''}$ $=46^\circ\,40'$	[36, 17, 46, 40-e 46, 40 íb-si₈ 3,]20 *ša tu-uš-ta-ki*[*-lu*]
$Z=46^\circ\,40'-3^\circ\,20'$ $=43^\circ\,20'$	6. *inside of* 46° 40′ *you tear out:* 43° 20′ *you inscribe* [*lìb-bi* 46, 40 *ta-na-sà-aḫ-*]*ma* 43, 20 *ta-la-pa-a*[*t*]
$1^\circ\,26'\,40''\cdot z=43^\circ\,20'$,	7. The igi of 1° 26′ 40″ *is not detached. What to* 1° 26′ 40″ [igi 1, 26, 40 *ú-la ip-pa-ṭ*]*a-ar mi-nam a-na* 1,2[6, 40]
$1^\circ\,26'\,40''\cdot 30=43^\circ\,20'$ whence $z=30$	8. *shall I pose which* 43° 20′ *gives me?* 30 *its bandûm*[b]. [*lu-uš-ku-un ša* 43, 20 *i-n*]*a-di-nam* 30 *ba-an-da-šu*
$x=1\cdot z=1\cdot 30=30$	9. 30 *to* 1 *you raise:* 30 the *first confrontation.* [30 *a-na* 1 *ta-na-ši-ma* 30] *mi-it-ḫar-tum iš-ti-a-at*
$y=40'\cdot z+5$ $=40'\cdot 30+5$	10. 30 *to* 40′ *you raise:* 20; *and*[c] 5 *you append:* [30 *a-na* 40 *ta-na-ši-ma* 20] *ú* 5 *tu-ṣa-ab-ma*
$=20+5=25$	11. 25 the *second confrontation.* [25 *mi-it-ḫar-t*]*um ša-ni-tum*

[a] From obv. II, 47 to rev. I, 5, only a few signs are preserved; from rev. I, 6 to 11, c. half of each line is preserved. In spite of this, the reconstruction (due to Thureau-Dangin 1936a, taken over in MKT III, 3) appears to be subject to very little doubt, thanks to the closely related No 24 of the same tablet.

[b] Probably a Sumerian loanword (cf. AHw, 102); is it also found in rev. I, 35 of the same tablet, where the numerical value of the entity is $^1/_4$. The mathematical function of the term is obvious, the factor to be multiplied unto 1° 26′ 40″ if we are to obtain the product 43° 20′. The general meaning of the term is unclear, but could perhaps be "that which is to be given together with" (ba, "to allot" etc.; -da, comitative suffix <"side").

[c] Both F. Thureau-Dangin and O. Neugebauer interprete this passage as "20 and 5 you append". Only here, however, and in two strictly parallel passages (rev. II, 31 and 32) is "append" found together with an "and". It is obviously the "and 5 nindan" of obv. II, 45 which gives rise to the present "and" (while corresponding statements in rev. II, 18f. give rise to the other occurrences of the construction). This suggests the interpretation given here. The observation made in note c to VAT 8389 Nº 1 (section VII. 1) supports the interpretation, especially because the use of the agentive suffix -e after results in a number of places in the present tablet suggests that results are even here to be understood as nominatives (the natural Akkadian understanding of the Sumerian agentive, the subject case for transitive verbs only).

This calls for various observations. On the one hand the operations correspond precisely to those of a modern solution to the same problem, or to those of a Medieval rhetorical solution. The Babylonians were as fully able to reduce the problem to a basic type as were the Islamic algebrists or their more recent descendants, in spite of their concrete and geometric way of thought. On the other hand, the concrete and geometric method is present all the way through, not only in the final reduction of the basic problem $\alpha x^2+\beta x=\gamma$ (rev. I, 2–9). The squaring of $(40'\cdot x+5)$ appears to be imagined geometrically (cf. Fig. 11): $40'\cdot 40'$ and $5\cdot 5$ are made by "spanning", while the coefficient $5\cdot 40'$ (an operation of proportionality, replacing "5 confrontations" by "$(40'\cdot 5)$ confrontations") is performed as a "raising". Great care is taken to take the factor *1* into account and to square it (rev. I, 1 and 9); the reduction to basic type, finally, avoids the unnecessary step to find the total number of "confrontations", which anyhow would have to be bisected.

If we go a bit closer to the text, we notice that the problem is reduced to the basic type of BM 13901 No 3 (section V.5); but the unknown "confrontation" of this reduced problem is not identical with the greater "confrontation" of the problem. Instead, the two confrontations of the problem are 1 times this unknown and 40′ times the unknown plus 5, respectively (this is why the symbolic trans-

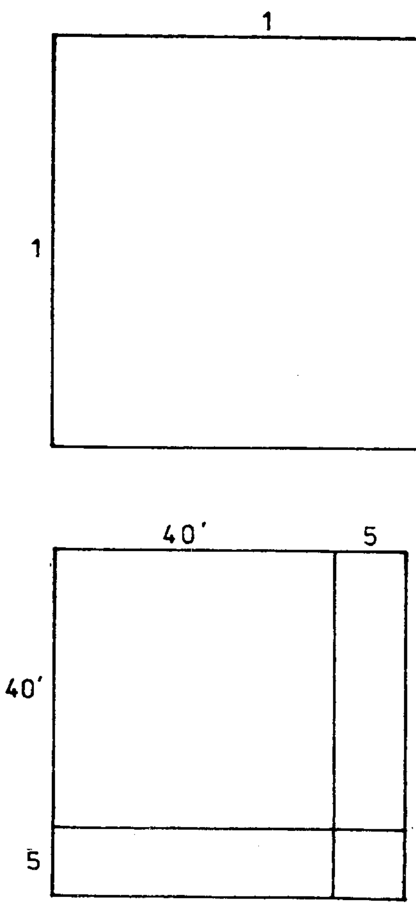


Figure 11. The two "confrontations" of BM 13901 N° 14, with 1, 40′ and 5 "inscribed", as stated in obv. II, 6.

lation in the left margin introduces a variable z). An analogous distinction between a "true width" and a "width" obtained through a multiplication by *1* could be found in TMS XVI A, line 10. In both cases, the distinction can be said to be a distinction between an original problem and its "basic representation". In the present case, as mostly when concrete entities are represented, the representing entities are not mentioned by any name; we can only see from the calculational steps that a specific basic type is dealt with (here that of N° 3 of the same tablet; cf. section V.5).

VIII.2. AO 8862 N^os^ 1–3 (MKT I, 108–111)

Like BM 13901, this tablet belongs to the earliest documented phase of Old Babylonian algebra. The first three sections deal with problems of essentially the same structure ($x+y=S$, $xy+\alpha x+\beta y=A$) and might have been solved slavishly by the same procedure. Instead, however, N^os^ 1 and 2 make use of the same principle but apply it differently, while N° 3 goes quite different ways. The three problems taken together thus constitute a fine demonstration of the flexibility of Babylonian algebraic procedures.–Had Babylonian mathematics been nothing but a collection of standardized recipes, everything on the tablet had looked differently.

N° 1 was also the first Babylonian algebraic text for which a geometrical explanation was given, viz. by K. Vogel as early as 1933.[136] Finally, the problems are interesting because of various details in the formulations. As these details can all be demonstrated on N^os^ 1–2, I restrict the translation to these two problems, and explain N° 3 only in symbolic and geometric interpretation.

N° 1	I
	1. Length, width[a]. Length *and* width *I have made span:* uš sag uš *ù* sag *uš-ta-ki-il*$_5$*-ma*
	2. a sur*face I have built* a-šà^{lam} *ab-ni-i*
	3. *I went around* (it). *So much as* length *over* width *as-sà-ḫi-ir ma-la* uš *e-li* sag
	4. *goes beyond* *i-te-ru-ú*
$x\cdot y+(x-y)=3'\ 3°$	5. *to* the *inside of* the sur*face I have appended* *a-na li-ib-bi* a-šà^{lim} *ú-ṣi-ib-ma*
	6. 3' 3°. *I turned back.* Length *and* width 3, 3 *a-tu-úr* uš *ù* sag
$x+y=27$	7. I have accumulated: 27. Length, width *and* surface *what?* gar-gar-*ma* 27 uš sag *ù* a-šà *mi-n[u-u]m*

136 Vogel 1933: 79, in a comment upon Neugebauer 1932a.

[b]

27	3' 3°	*things accumulated*
15	length	3' surface
12	width	
27	3, 3	*ki-im-ra-tu-ú*
15	uš	3 a-šà
12	sag	

8. *You, by your making,*
at-ta i-na e-pe-ši-i-ka

9. 27, the *things accumulated of* length *and* width
27 *ki-im-ra-at* uš *ù* sag

$$xy+(x-y)+(x+y) = xy+2x = x\cdot(y+2) = 3'\ 30^\circ$$

10. *to* the *inside of* 3' 3° *append:*
a-na li-bi [3, 3] *ṣi-ib-ma*

11. 3' 30°. 2 *to* 27 *append:*
3, 30 2 *a-na* 27 *ṣi-ib-ma*

$$x+(y+2)=29$$

Putting $Y=y+2$:

$$xY=3'\ 30^\circ,\ x+Y=29$$

12. 29. *Its* MOIETY, *that of* 29, *you break:*
29 BA.A-*šu š*[*a*] 29 *te-ḫe-ep-pe-e-ma*

$$\left(\frac{x+Y}{2}\right)^2=14^\circ\ 30'^2 = 3'\ 30^\circ\ 15'$$

13. 14° 30′ steps of 14° 30′, 3' 30° 15′.
14, 30 a-rá 14, 30 3, 30, 15

$$\left(\frac{x-Y}{2}\right)^2=\left(\frac{X+Y}{2}\right)^2-xY = 15'$$

14. *From* the *inside of* 3' 30° 15′
i-na li-bi 3, 30, 15

15. 3' 30° *you tear out:*
3, 30 *ta-na-sà-aḫ-ma*

$$\frac{x-Y}{2}=\sqrt{15'}=30'$$

16. 15′ the *remainder*. 15′ makes 30′ equilateral
15 *ša-pi-il₅-tum* 15-e 30 íb-[si₈]

$$x=\frac{x+Y}{2}+\frac{x-Y}{2} = 14^\circ\ 30'+30'=15$$

17. 30′ *to* the *first* 14° 30′
30 *a-na* 14, 30 *iš-te-en*

18. *append:* 15 the length.
ṣi-ib-ma 15 uš

$$Y=\frac{x+Y}{2}-\frac{x-Y}{2} = 14^\circ\ 30'-30'=14$$

19. 30′ *from* the *second* 14° 30′
30 *i-na* 14, 30 *ša-ni-i*

20. *you cut off:* 14 the width
ta-ḫa-ra-aṣ-ma 14 sag

21. 2 *which to* 27 *you have appended*
2 *ša a-na* 27 *tu-uṣ₄-bu*

$$y=Y-2=12$$

22. *from* 14, the width, *you tear out:*
i-na 14 sag *ta-na-sà-aḫ-ma*

23. 12 the true width.
12 sag gi-na

Proof:

$$x\cdot y=15\cdot 12=3'$$

24. 15, the length, 12 the width, *I have made span:*
15 uš 12 sag *uš-ta-ki-il₅-ma*

25. 15 steps of 12, 3' the surface.
15 a-rá 12 3 a-šà

26. 15, the length, *over* 12, the width,
15 *uš e-li* 12 sag

$$x-y=3$$

27. by *what goes beyond?*
mi-na wa-ta-ar

$$xy+(x-y)=3'+3 = 3'\ 3^\circ$$

28. 3 *it goes beyond*; 3 *to* the *inside of* 3', the surface, *append.*
3 *i-te-er* 3 *a-na li-bi* 3 a-šà *ṣi-ib*

29. 3' 3° the surface.
3, 3 a-šà

[a] F. Thureau-Dangin translated "length, width" (uš sag) simply as "rectangle" (e.g. TMB, 64). That this is indeed the correct interpretation of the composite expression is confirmed by the Susa table of constants (TMS III, 32), which speaks of the "diagonal of length and width", meaning the diagonal of a standard rectangle of sides 45′ and 1.

[b] This arrangement of the statement between lines 7 and 8 follows the autography (MKT II, plate 35).

N° 2

I

30. Length, width. Length *and* width
uš sag uš *ù* sag

31. *I have made span:* A surface *I have built.*
uš-ta-ki-il₅-ma a-šà*[lam]* *ab-ni*

32. *I went around* (it). The *half of* the length
a-sà-ḫi-ir mi-ši-il₅ uš

33. *and* the *third of* the width
ù ša-lu-uš-ti sag

34. *to* the *inside of my* surface
a-na li-bi a-šà-*ia*

$$xy+{}^1\!/_2x+{}^1\!/_3y=15$$

35. *I have appended:* 15.
[*ú*-]*ṣi-ib-ma* 15

36. *I turned back*. Length *and* width
[*a-t*]*u-úr* uš *ù* sag

$$x+y=7$$

37. *I have accumulated:* 7.
[*ak*-]*mu-ur-ma* 7

II

1. Length *and* width *what?*
uš *ù* sag *mi-nu-um*

2. *You, by your making,*
at-ta i-na e-pe-ši-i-ka

3. 2 (as) *inscription of* the *half*
[2 *n*]*a-al-p*[*a*]*-at-ti mi-iš-li-im*

4. *and* 3 (as) *inscription*
[*ù*] 3 *na-al-pa-ti*

	5. *of* the *third you inscribe:* [*ša-*]*lu-uš-ti ta-l*[*a*]*-pa-at-ma*
	6. The igi of 2, 30′, *you detach:* igi 2-bi 30 *ta-pa-ṭar-ma*
$^1/_2 \cdot (x+y) = 3°\,30'$	7. 30′ steps of 7, 3° 30′; *to* (the place of) 7, 30 a-rá 7 3, 30 *a-na* 7
	8. (of) the *things accumulated*[a], length *and* width, *ki-im-ra-tim* uš *ù* sag
	9. *I bring:* *ub-ba-a*[*l*]*-ma*
$xy + {}^1/_2 x + {}^1/_3 y - {}^1/_2(x+y)$ $= xy - ({}^1/_2 - {}^1/_3)\, y$ $= 11°\,30'$	10. 3° 30′ *from* 15, *my things accumulated* 3, 30 *i-na* 15 *ki-i*[*m*]*-ra-ti-i-a*
	11. *cut off:* *ḫu-ru-uṣ*$_4$*-ma*
	12. 11° 30′ the *remainder.* 11, 30 *ša-pi-il*$_5$*-tum*
${}^1/_2 - {}^1/_3 = 1/(2 \cdot 3)$ $= 1/6 = 10'$	13. *Go not beyond.* 2 *and* 3 *I make span:* *l*[*a*] *wa-t*[*ar*] 2 *ù* 3 *uš-ta-kal-ma*
	14. 3 steps of 2, 6. 3 a-rá 2 6
Putting $X = x - 10'$ we have $Xy \quad = 11°\,30'$ $X + y = 7 - 10' = 6°\,50'$	15. The igi of 6, 10′ *it gives you.* igi 6 gál 10 *i-na-di-kum* 16. 10′ *from* 7, *your things accumulated*[b] 10 *i-na* 7 *ki-im-ra-ti-i-ka*
	17. of length *and* width *I tear out:* uš *ù* sag *a-na-sà-aḫ-ma*
	18. 6° 50′ the *remainder.* 6, 50 *ša-pi-il*$_5$*-tum*
$\frac{X+y}{2} = 3°\,25'$	19. *Its* MOIETY, *that of* 6° 50′, *I break:* BA.A-*š*[*u*] *ša* 6.50 *e-ḫe-pe-e-ma*
	20. 3° 25′ *it gives you.* 3, 25 *i-na-di-ku*
	21. 3° 25′ *until twice* 3, 25 *a-di ši-ni-šu*
$\left(\frac{X+y}{2}\right)^2 = 11°\,40'\,25''$	22. *you inscribe:* 3° 25′ steps of 3° 25′, *ta-la-pa-at-ma* 3.25 a-rá 3, 25
$\left(\frac{X-y}{2}\right)^2 = \left(\frac{X+y}{2}\right)^2 - Xy$ $= 10'\,25''$	23. 11° 40′ 25″; *from* the *inside* 11, 40, [25] *i-na li-bi*
	24. 11° 30′ *I tear out* 11, 30 *a-na-sà-aḫ-ma*
$\frac{X-y}{2} = \sqrt{10'\,25''} = 25'$	25. 10′ 25″ the *remainder.* ⟨10′ 25″ makes 25′ equilateral⟩ 10, 25 *ša-pi-il*$_5$*-tum* ⟨10′ 25″-e 25′ íb-si$_8$⟩

$X = \frac{X+y}{2} + \frac{X-y}{2}$ $= 3°\,25' + 25' = 3°\,50'$	26. *To* the *first* 3° 25′ *a-na* 3, 25 *iš-te-en*
	27. 25′ *you append:* 3° 50′. 25 *tu-ṣa-am-ma* 3, 50
$x = X + 10'$ $= 3°\,50' + 10' = 4$	28. *and* (that) *which from* the *things accumulated of* *ù ša i-na ki-im-ra-at*
	29. length *and* width *I have torn out* uš *ù* sag *a*[*s*]*-sà-aḫ-ma*
	30. *to* 3° 50′ *you append:* *a-na* 3, 50 *tu-ṣa-am-ma*
$y = \frac{X+y}{2} + \frac{X-y}{2}$ $= 3°\,25' - 25' = 3$	31. 4 the length. *From* the *second* 3° 25′ 4 uš *i-na* 3, 25 *ša-ni-im*
	32. 25′ *I tear out:* 3 the width. 25 *a-na-sà-aḫ-ma* 3 sag
	32a.[c] 7 the *things accumulated* 7 *ki-im-ra-tu-ú*

32b.	4 length 3 width	12	surface
	4 uš 3 sag	12	a-šà

[a] Since *kimrātum* is written in the status rectus (*ki-im-ra-tim*) and not in status constructus, "length and width" must stand (in this single case) as an apposition, not as the second member of a genitive construction. Hence the translation.

[b] In most of its occurrences, *kimrātum* stands so that it cannot be decided whether a (most peculiar) singular feminine *kimratum* or a plural *kimrātum* is meant. The indubitable plural of II, 32a could at a pinch be explained away (F. Thureau-Dangin, TMB, 67, does so, translating "7 ⟨et 15⟩, les sommes"). In II, 16, however, there can be no doubt that a single sum is spoken of in the plural, as *ki-i*[*m*]*-ra-ti-i-ka*. The *ki-i*[*m*]*-ra-ti-i-a* of II, 10 is also a most certain plural.
It is noteworthy that the singular form to be expected from the plural (*kimirtum*) is completely absent from the texts. It appears to be established beyond reasonable doubt that the single sum is designated by the plural form (and hence to the plurality of addends), as presupposed in my standard translation.

[c] This ordering follows the autography (MKT II, plate 36). There is no doubt that 32a is meant as a separate line, while the rest (32b) stands as a tabulation.

Designating as usual the length as x and the width as y we can finally transcribe problem 3 as follows:

$$xy + (x-y)\,(x+y) = 1''\,13'\,20° \qquad x + y = 1'\,40°$$

and from the way the solution is formulated is is clear that the author was aware that this was equivalent to

$$x+y=1'\ 40^\circ \qquad xy+1'\ 40^\circ\cdot(x-y)=1''\ 13'\ 20^\circ$$

which could easily be reduced to a standard problem $Xy=A$, $X+y=B$ by the method already known from N^{os} 1–2. Instead, however, the following steps occur:

$$(x+y)^2=2''\ 46'\ 40^\circ$$
$$(x+y)^2-xy-(x+y)\ (x-y)=1''\ 23'\ 20^\circ$$

which, putting $x+y=1'\ 40^\circ=a$, reduces to

$$y^2+ay=1''\ 33'\ 20^\circ, \text{ whence}$$
$$\left(y+\frac{a}{2}\right)^2=1''\ 33'\ 20^\circ+(1'\ 40^\circ/2)^2=2''\ 15'$$
$$y+\frac{a}{2}=y+\frac{x+y}{2}=\sqrt{2''\ 15'}=1'\ 30^\circ$$
$$\frac{x-y}{2}=(x+y)-\left(y+\frac{x+y}{2}\right)=1'\ 40^\circ-1'\ 30^\circ=10$$

and so finally

$$x=\frac{x+y}{2}+\frac{x-y}{2}=50+10=1'$$
$$y=\frac{x+y}{2}-\frac{x-y}{2}=50-10=40$$

– all of it formulated of course the usual way. The procedure is fully correct, but it looks rather queer in the above symbolic transcription.

First of all the construction of the three problems should be noted. Invariably, a surface is "built", after which the teacher "goes around". As A. Westenholz first suggested to me the text looks like a tale about real surveying: The teacher-surveyor marks out a field (the everyday meaning of a-šà and *eqlum*, we remember) in the terrain, after which he goes around it, pacing off its measures. Only after this walk, indeed, do numbers enter the text, as if, e.g., the excess of length over width is only known now. Using his newly acquired knowledge, the surveyor joins some extra areas to the field–"appending", we observe, not "accumulating" as when measures of sides and surfaces were added in BM 13901. This must of course be done in the terrain, from which he then turns back in order to state the sum ("accumulation") of length and width.

After this observation we shall look at the procedures which appear to be used to solve the three problems. The steps of problem 1 can be easily followed on Fig. 12. The simple addition of one length and one width (regarded as rectangles of width *1*, which is not said explicitly) transforms the irregular surface of area 3' 3° into a rectangle of which the area and the sum of length (x) and width (Y) are known. A bisection of this known length $x+Y=29$, to which the rectangle $x\cdot Y$ is "applied with defect", allows us to reconstruct the rectangular area as a gnomon. The area and hence the side of the small square enclosed by this gnomon are found, and the original dimensions of the rectangle $x\cdot Y$ follow as usual. In this way, everything labelled "length", "width" or "surface" is indeed a length, a width or a surface.

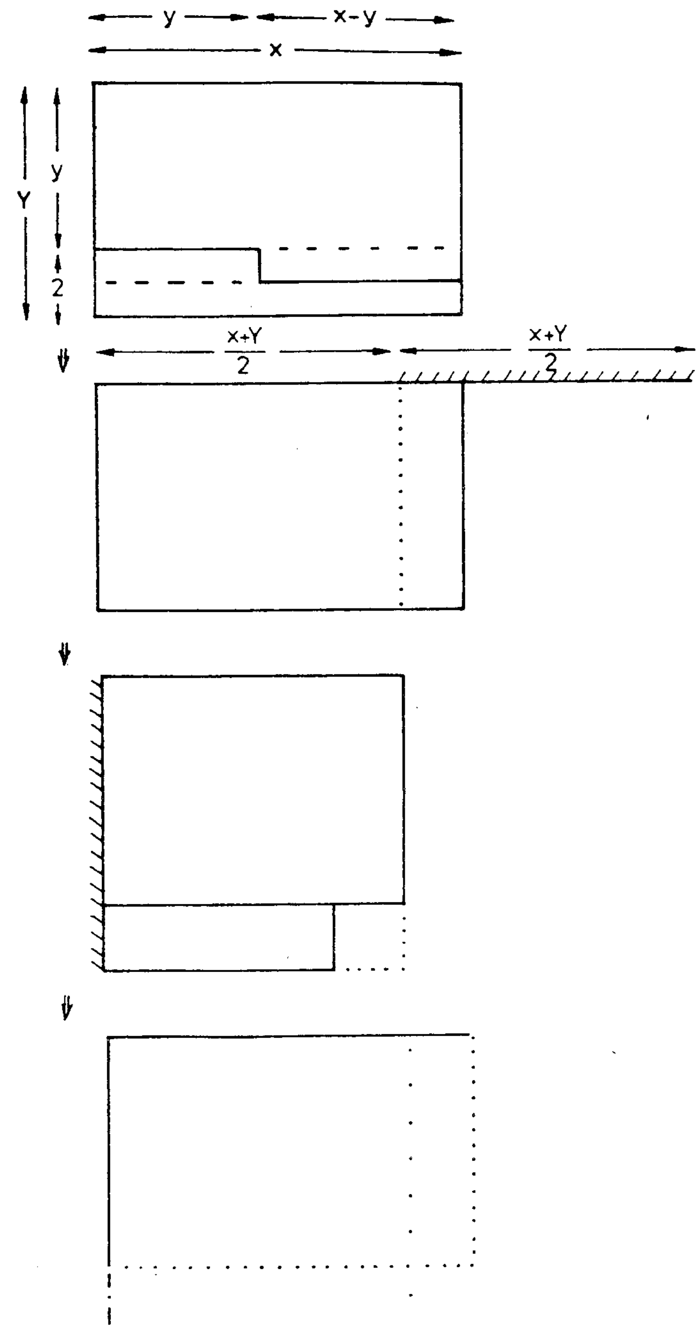


Figure 12. The geometrical interpretation of AO 8862 N° 1. Distorted proportions.

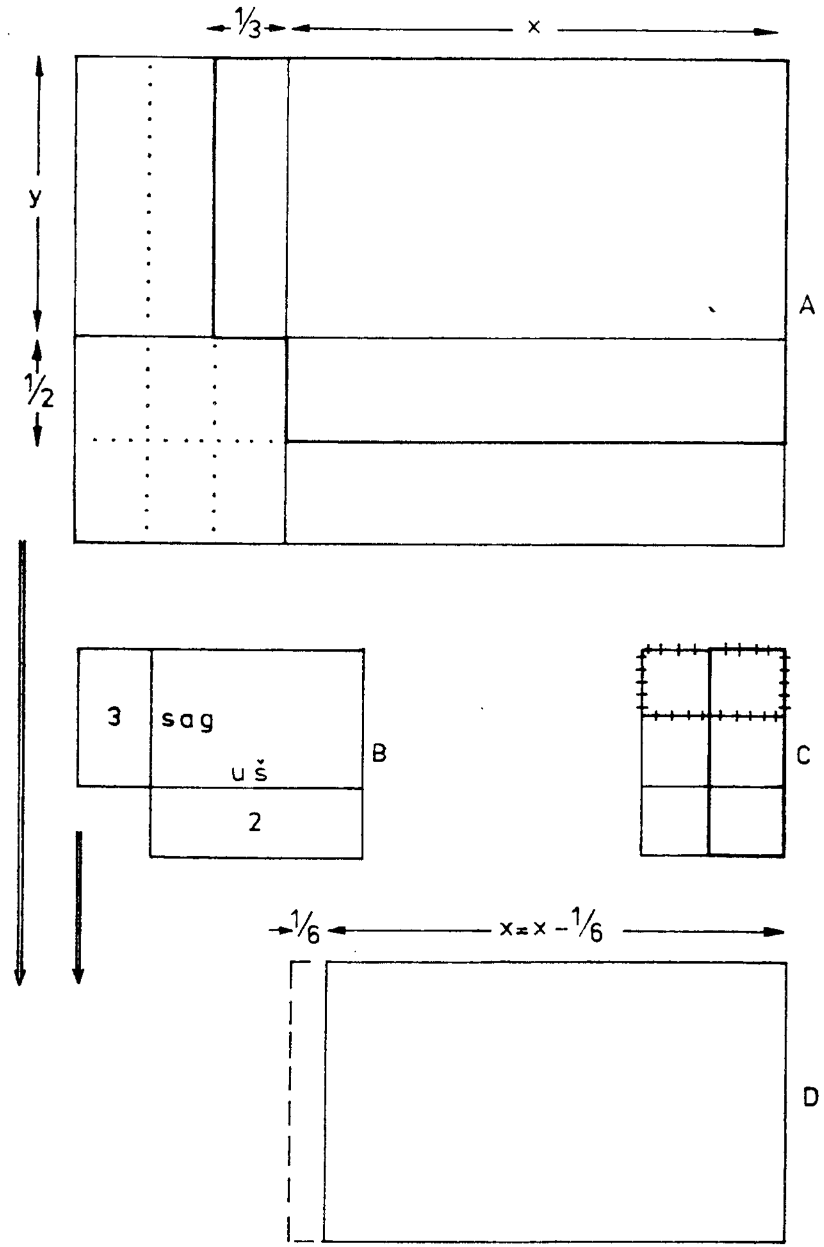


Figure 13. The geometrical interpretation of AO 8862 N° 2. Distorted proportions.

We observe that the procedure is different from the one shown on Figures 4–6, which corresponded to "application with excess". The corresponding problem in one variable is the type $\alpha x - x^2 = \beta$–to give it a formulation which could be formulated inside the Babylonian framework: "from α confrontations I have torn out the surface: γ". This is the type which has two positive solutions; it seems to be completely absent from the Babylonian material[137] even though the corresponding problem in two variables is very common.

The reduction of N° 2 is somewhat more complex, but follows the same pattern, see Fig. 13. Fig. 13A shows the configuration as we would imagine the geometric situation described, while Fig. 13B describes what appears to correspond more or less to the Babylonian understanding, as described in the text. The numbers *2* and *3* are "inscribed as inscriptions" of $^1/_2$ and $^1/_3$, probably along the edges of the rectangle, to remind that the widths of these edges are to be understood, not as *1* but as stated; and when $^1/_2x + {}^1/_2y$ is to be subtracted from the aggregated surface it is "brought to" the place of "length and width", viz. to those entities which were accumulated. It is indeed clear from the text that the 3° 30′ is not brought to an abstract sum (which would also be mathematically meaningless) but to the collection of added yet still separate entities–a point where the plural and hence concrete character of *kimrātum* is of importance.

When the half-sum of length and width is brought to the place of length and width, i.e. to the edges of the rectangle, it is obvious and not commented upon that the $^1/_2$-length is eliminated; but more than $^1/_3$-width goes away, and a curious calculation in II.13–15 finds the resulting defect to be 10′ (width). The process of "making 2 and 3 span" can be imagined as in the lower left corner of Fig. 13A; but an independent procedure as shown in Fig. 13C seems more plausible, among other things because of the explicit order to stop the ongoing procedure and because Fig. 13A is described as a real field in the terrain. In sort of parenthesis, an entity is "built" of which both $^1/_2$ and $^1/_3$ are easily taken, to allow for a two-dimensional variant of the "single false position"" (cf. below).

From here on, everything runs as in No 1.

The geometrical reading of N° 3 is shown in Fig. 14. It turns out that the squaring of $x+y$ gives us a figure from which the given surface $xy+(x-y)\,(x+y)$ can easily be torn out. The figure is seen to be of precisely the same structure as that shown in Fig. 2, and other texts suggest that it was familiar in the Old Babylonian period too.[138] What remains is a square of side y and a rectangle of sides y and $x+y$. This remainder is easily rearranged as a gnomon, as done in Fig. 14B. The usual quadratic completion yields a side of the completed square equal to 1` 30°.

If the rearrangement had been thought of as a problem in y (the sag), $y^2+50\cdot y = 1``\,33`\,20°$, then it might have been natural to subtract 50 from this 1` 30° $(=y+50)$. Instead, however, 1` 30° is subtracted from the side of

137 Absent, that is, in explicit formulation. Indications exist, indeed, that the problem BM 85194 rev. II 7–21 was solved as a problem in one variable and not in two, as it was once proposed by Vogel 1936: 710. See my 1985: 59f.

138 So YBC 6504 No 2 (MKT III 22, interpretation in my 1989: 28–31) and BM 13901 No 19 (MKT III 4). In both cases, the linear dimensions of the figure are half of those of the present problem (30 and 20, against 1` and 40°).

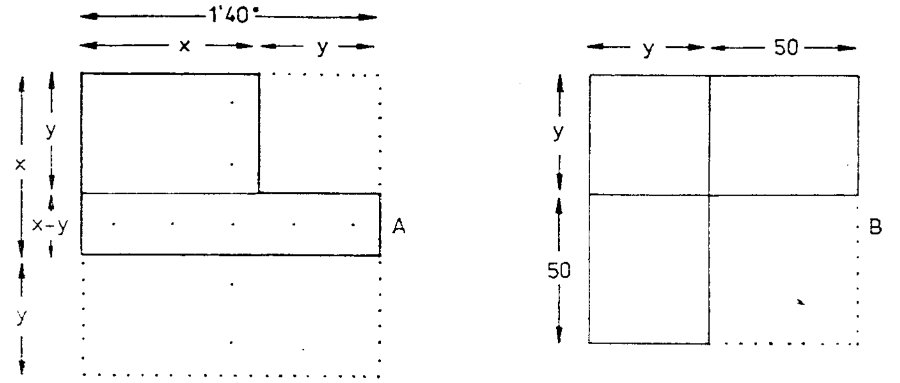


Figure 14. The geometrical interpretation of AO 8862 N° 3.

the square of Fig. 14A. If we look at the subdivision of this square through the quartering lines it is indeed evident that the difference between the two entities is the half-difference between the length and the width of the original rectangle. It seems thus as if the steps shown in Fig. 14B shall not be apprehended as a change of problem; instead, everything is to be understood all the way through in terms of the constituent parts of Fig. 14A. By extension, we may surmise that the "changes of variable" to Y and X in Nos 1 and 2 are not really to be understood as explicit changes of the unknown. That is indeed a comprehension inspired by rhetorical or symbolic algebra where certain entities are distinguished by having a name of their own and are hence regarded as fundamental unknowns. Instead, all entities in a figure which are not known are unknown on an equal footing as far as the solving procedure is concerned. Only as far as certain entities are asked for initially can they be considered privileged (and relatively privileged only, as the entities asked for in the beginning and those found in the end need not coincide[139]). This corresponds to our own comprehension of problems of geometrical analysis–the phrase to be understood in its Greek sense.

A number of features of the texts call for separate discussion. Most important among these is the occurrence of the term a-rá, "steps of", the multiplicative term of the multiplication tables. In some places it stands alone, but time after other it is found in double constructions that show the isolated occurrences to be ellipses. Other texts state that a rectangle is to be built from a length and a width, and leave the numerical multiplication implicit, giving directly its result.[140] In the present double constructions, both steps are spelled out explicitly, the multiplication apparently through reference to the auxiliary tables, and in I, 13 and in two places in No 3, it is the building process which is left implicit.[141]

[139] Such a discrepancy is found, e.g., in BM 85194 rev. II 7–21.

[140] Similarly, we remember, the "raising" was sometimes left implicit in the "posing" of one number to another (above, section IV.6).

[141] It may be significant that two of the three ellipses occur after the "breaking" of a "moiety", which already may imply the construction process; similarly, indeed, in II 21, the moieties are not "made span" but instead "inscribed until twice". The third

Another terminological peculiarity of the text is the use of the subtractive term *ḫarāṣum*, "to cut off", along with the more current *nasāḫum*, "to tear out". Already from the metaphorical contents of the two terms we migth expect that the latter would be preferred for identity-conserving subtraction from surfaces and the former for the shortening of one-dimensional entities, if a distinction were to be made. This is, indeed, precisely the main tendency of this as well as all other texts where the terms are found together. But it is only a tendency, in the sense that *nasāḫum* may be used for one-dimensional entities too; most clearly this is seen in I, 19–22: First 30′ is "cut off" from 14° 30′, and next 2 is "torn out" from the resulting 14.[142] It is thus excluded to regard the two terms as names for distinct operations. At the same time the tendential distinction prevents us from seeing the terms as connotationally neutral technical terms, whose metaphorical basis had been completely worn off. They constitute instances of mathematical terms which must be "regarded as open-ended expressions which in certain standardized situations are used in a standardized way" (as formulated above, note 29).

A third formulation of interest is the recurrent BA.A-*šu ša*, "its moiety, that of", which is found in all three problems at the point where a rectangle is bisected in order to allow a gnomonic reorganization (I, 12; II, 19; III, 13). The use of the determinative pronoun *ša* shows that the quantity pointed at, the one which is to be bisected, must have some independent existence, mental or physical, which allows us to think of or point at a definite entity. I, 12, for instance, cannot be read as the bisection of an abstract number *29*; it must by necessity deal with something definite–another confirmation of the concreteness inherent in the naive-geometric interpretation.

A final terminological point to be observed is the distinction which is maintained between *mišlum*, "half", and *bāmtum*, "moiety", and the corresponding distinction between multiplication by igi 2-bi = 30′ (N° 2, II, 6) and "breaking". Once more "breaking" is seen to be reserved to describe bisection into natural "wings" (cf. section IV.5, and note b to BM 13901 N° 1, section V.2).

As concerns the mathematical aspect of the texts, the flexible handling of problems and methods was already pointed at in the introductory remarks. It makes clear that the understanding behind the text must have been flexible, too, that it has nothing to do with blind application of fixed rules or algorithms discovered by equally blind luck, as claimed too often in the secondary literature.

Another related implication of the tablet concerns the purpose of such texts. I think of the tabulation between I, 7 and I, 8. Here, before the description of the solving procedure, the whole construction and solution of problem 1 is told

ellipsis, finally, is found when the area of the square in Fig. 17A is found: If this configuration is well-established beforehand, there is no need to construct it anew (cf. the concluding discussion in section V.8).

[142] But if we look at the written numbers, the distinction holds good even in this case, as A. Westenholz has observed: When 30′ is removed from 14° 30′ it is the end of the number (viz. of the sequence 10,4,30) which is "cut off"; but to take away 2 from the sequence 10,4 requires that we remove part of the compact group of wedges making up the 4.

In one text, viz. YBC 4675 obv. 14, is *ḫarāṣum* used to designate a subtraction from a surface (4′ 49°–2′). That text, however, avoids *nasāḫum* altogether.

*22

in advance. The subsequent procedural prescriptions can therefore hardly be seen as an attempt to find the unknown dimensions of the rectangle. The aim is not really to solve the problem and find the solution; it is to demonstrate how to solve the problem, to present an argued solution.

The calculation in No 2, II, 13–15, finally, is remarkable, though belonging more on the level of details. The Babylonian predilection for argumentation by means of a "single false position" was pointed out repeatedly above in sections V.6 and especially VII.3, where a representation by countable units was also suggested. Here, however, the trick is extended into two dimensions, as revealed by the term "making span" (extension apart, its relation to the calculation of $1-\frac{1}{4}=45'$ in TMS XVI is obvious). Since $\frac{1}{6}$ is stated directly to be 10′, the identities $\frac{1}{2}=30'$ and $\frac{1}{3}=20'$ can hardly have been considered a secret. The computation of their difference by way of a geometrical subtlety must therefore be seen as a didactical nicety, as a means to demonstrate the extension of the simple argument.

VIII.3. TMS IX (TMS, 63f.; cf. von Soden 1964)

Such didactical concerns are even more obvious in the Susa text TMS IX, which approaches the style of TMS XVI (above, section VII.3). In this case, however, the text goes from simplest ($xy+x=40'$) to less simple ($xy+x+y=1$) fundamental equation, ending with a fairly complex application of the fundamental principle.

Unfortunately, the transcription in TMS is not very precise, the restitution of damaged lines and the translation are worse, and the mathematical commentary is at times nonsensical. Had it not been for these circumstances, the text would probably have changed much conventional wisdom in the understanding of Babylonian mathematics 25 years ago.

PART A $(x=30', y=20')$ $x\cdot y+1\cdot x=40'$	1. The surface *and* 1 length ACCUMULATED, 40′. [(30′ the length 20′ the width)[b]][a] a-šà *ù* 1 uš UL.GAR 4[0 (30 uš 20 sag)]
Alternative approaches to an understanding:	2. *As* 1 length *to* 10′, the surface [has been appended][a] *i-nu-ma* 1 uš *a-na* 10 ⌜a-šà daḫ]
$Y=y+1=20'+1°$ $=1°20'$ or	3. *Either* 1 as BASE(?)[c] *to* 20′, the width, [append] *ú-ul* 1 KI.GUB.GUB *a-na* 20 [sag daḫ]
$x\cdot 1°20'=40'$	4. *or* 1° 20′ *to* the width *which* 40′ *together with* [the length (SURROUNDS pose)][a] *ú-ul* 1, 20 *a-na* sag *šà* 40 *it-*⌜*ti* uš (NIGIN gar)]
or $1°20'\cdot 30'=40'$	5. *or* 1° 20′ *together with* 30′ the length MAKE SURROUND, 40′ *its name* *ú-ul* 1, 20 *it-⟨ti⟩* 30 uš NIG[IN] 40 *šum-[šu]*
Implicit conclusion: $x\cdot y+1\cdot x=x\cdot(y+1)$	6. *Since so, to* 20′, the width, *which has been said to you* *aš-šum ki-a-am a-na* 20 sag *šà qa-bu-ku*

	7. 1 is appended: 1° 20′[d] *you see. Out from here* 1 daḫ-*ma* 1, 20 *ta-mar iš-tu an-ni-ki-a-am*
	8. *you ask.* 40′ the surface, 1° 20′ the width, the length *what?* *ta-šà-al* 40 a-šà 1, 20 sag uš *mi-nu*
	9. [30 the length][a]. *So* the *having-been-made* [30 uš *k*]*i-a-am ne-pé-šum*
PART B $(x=30', y=20')$ $x\cdot y+x+y=1$	10. [Surface, length *and* width AC][a] CUMULATED, 1. *By* the *Akkadian* [a-šà uš *ù* sag U]L.GAR 1 *i-na ak-ka-di-i*
$(x+1)\cdot(y+1)$ $=x\cdot y+1\cdot x+1\cdot y+1\cdot 1$	11. [1 *to* the length append.][a] 1 *to* the width append. *Since* 1 *to* the length is appended, [1 *a-na* uš daḫ] 1 *a-na* sag daḫ *aš-šum* 1 *a-na* uš daḫ
$1\cdot 1=1$, and so	12. [1 *to* the width is app][a]ended, 1 *and* 1 MAKE SURROUND, 1 *you see.* [1 *a-na* sag d]aḫ 1 *ù* 1 NIGIN 1 *ta-mar*
$(x+1)\cdot(y+1)$ $=(x\cdot y+x+y)+1$ $=1+1=2$	13. [1 *to* the ACCUMULATION of length,][a] width *and* surface append, 2 *you see* [1 *a-na* UL.GAR uš] sag *ù* a-šà daḫ 2 *ta-mar*
$Y=y+1=1°20'$ $X=x+1=1°30'$	14. [(*To* 20′ the width 1 appe)][a]nd, 1° 20′. *To* 30′ the length 1 append, 1° 30′. [(*a-na* 20 sag 1 da)]ḫ! 1, 20 *a-na* 30 uš 1 daḫ 1, 30
$X\cdot Y=1°30'\cdot 1°20'$	15. [(*Since* a surfa)][a]ce, *that of* 1° 20′ the width, *that of* 1° 30′ the length [(*aš-šum* a-š)]à *šà*! 1, 20 sag *šà* 1, 30 uš
	16. [(Length *together with* wid)][a]th *is made span*[e], *what is its name?* [(uš *it-ti* sa)]g! *šu-ta-ku-lu mi-nu šum-šu*
$X\cdot Y=2$	16a. 2 the surface 2 a-šà
	17. *So* the *Akkadian* *ki-a-am ak-ka-du-ú*
$x\cdot y+x+y=1$ PART C $y+\frac{1}{17}(3x+4y)=30'$	19. Surface, length *and* width ACCUMULATED, 1 the surface, 3 lengths, 4 widths ACCUMULATED, a-šà uš *ù* sag UL.GAR 1 a-šà 3 uš 4 sag UL.GAR
	20. *its* 17*th to* the width appended, 30′. [17]-*ti-šu a-na* sag daḫ 30
$17y+3x+4y=17\cdot 30'$ $=8°30'$	21. You, 30′ *to* 17 *go:* 8° 30′ *you see* [za-]e 30 *a-na* 17 *a-li-ik-ma* 8, 30 [*t*]*a-mar*
$17y+4y=21y$	22. *To* 17 widths, 4 widths append: 21 *you see,* [*a-na* 17 sag] 4 sag daḫ-*ma* 21 *ta-mar*
The coefficient of y is 21,	23. 21 *as much as* of widths, pose. 3 *of three of* lengths, [21 *ki-*]*ma* sag gar 3 *ša-la-aš-ti* uš

that of x is 3	24. 3 *as much as* of lengths, pose. 8° 30′ *what* is *its name?* [3 *ki*]-*ma* uš gar 8, 30 *mi-nu šum-šu*
$3\cdot x+21\cdot y=8^\circ 30'$	25. 3 lengths *and* 21 widths ACCUMULATED [3] uš *ù* 2[1 sa]g UL.[GAR]
	26. 8° 30′ *you see*[f] 8, 30 *ta-mar*
	27. 3 lengths *and* 21 widths ACCUMULATED [3] uš *ù* 21 sag UL.[GAR]
$x+1=X$ $y+1=Y$	28. 1 *to* the length append *and* 1 *to* the width append, MAKE SURROUND: [1 *a-na*] uš daḫ [*ù* 1 *a*]-*na* sag daḫ NIGIN-*ma*
$X\cdot Y=(xy+x+y)+1$ $=2$	29. 1 *to* the ACCUMULATION of surface, length *and* width append, 2 *you see*, 1 *a-na* UL.GAR a-šà uš *ù* sag daḫ 2 *ta-⟨mar⟩*
	30. [2 the sur]^a face. *Since* length *and* width, *those* of 2 the surface, [2 a-]šà *aš-šum* uš *ù* sag *šà* 2 a-šà
$X\cdot Y=1^\circ 30'\cdot 1^\circ 20'$ (identifications)	31. [1° 30′ the length *toge*]^a*ther with* 1° 20′ the width *is made span* [1, 30 uš *it*'$_1$-*ti* 1, 20 sag *šu-ta-ku-lu*
$1\cdot 1=1$	32. 1 the *appended*[g] *of* the length *and* 1 the *appended of* the width [1 *wu-ṣú*-]*bi* uš *ù* 1 *wu-ṣú-bi* sag
$1+(xy+x+y)=2$	33. [MAKE SURROUND, (1 *you see*). 1 *and* (. . .?)]^a the various (things)[h] ACCUMULATE, 2 *you see*. [NIGIN (1 *ta-mar*?) 1 *ù* (. . .?)] ḪI.A UL.GAR 2 *ta-mar*
$3X+21Y$ $=3+21+(3x+21y)$ $=3+21+8^\circ 30'=32^\circ 30'$	34. [(3, 21 *and* 8° 30′ ACCUMULATE)]^a, 32° 30′ *you see*. [(3 (. . .?) 21 (. . .?) *ù* 8, 30 (. . .?) UL.GAR] 32, 30 *ta-mar*
	35. *So you ask* [*ki-a*]-*am ta-šà-al*
$\tilde{y}=21Y$	36. [. . .] *of* the width *to* 21 ACCUMULAT(E/ION):[i] [. . .].TI sag *a-na* 21 UL.GAR-*ma*
$\tilde{x}=3X$	37. . . .[j] *to* 3, the lengths, *raise*, [. . .]ḪI(?) .A *a-na* 3 uš *i-ší*
$\tilde{x}\cdot\tilde{y}=3\cdot 21\cdot XY$ $=1ˋ 3^\circ\cdot XY$ $=1ˋ 3^\circ\cdot 2=2ˋ 6^\circ$	38. [1ˋ 3° *you see*. 1ˋ 3° *t*]^a*o* 2, the surface, *raise:* [1, 3 *ta-mar* 1, 3 *a*]-*na* 2 a-šà *i-ší-ma*
$\tilde{x}\cdot\tilde{y}=2ˋ 6^\circ$ $\tilde{x}+\tilde{y}=32^\circ 30'$ $\frac{\tilde{x}+\tilde{y}}{2}=16^\circ 15'$	39. [2ˋ 6° *you see* (2ˋ 6° the surface?)]^a 32° 30′ the ACCUMULATION *break*, 16° 15′ *you see*. [2, 6 *ta-mar* (2, 6 a-šà?)] 32, 30 UL.GAR *ḫi-pí* 16, 15 *ta-⟨mar⟩*
$\left(\frac{\tilde{x}+\tilde{y}}{2}\right)^2=(16^\circ 15')^2$ $=4ˋ 24^\circ 3' 45''$	40. {1[6° 15′ *you*]^a *see*}[k] 16° 15′ the counterpart pose; MAKE SURROUND, {16, 15 *ta-mar*} 16, 15 gaba gar NIGIN

	41. 4ˋ 24° 3′ 45″ *you see*. 2ˋ 6° *xxx*[l] 4, [24,]3, 45 *ta-mar* 2, 6 [. . .]
$\left(\frac{\tilde{y}-\tilde{x}}{2}\right)^2=\left(\frac{\tilde{y}+\tilde{x}}{2}\right)^2-\tilde{x}\tilde{y}$ $=2ˋ 18^\circ 3' 45''$	42. *from* 4ˋ 24° 3′ 45″ tear out, 2ˋ 18° 3′ 45′ *you see*. *i-na* 4, [2]4, 3, 45 zi 2, 18, 3, 45 *ta-mar*
$\frac{\tilde{y}-\tilde{x}}{2}=\sqrt{2ˋ 18^\circ 3' 45''}$ $=11^\circ 45'$	43. *What* it makes equilateral? 11° 45′ it makes equilateral. 11° 45′ *to* 16° 15′ append, *mi-na* íb-si 11, 45 íb-si 11, 45 *a-na* 16, 15 daḫ
$\tilde{y}=\frac{\tilde{y}+\tilde{x}}{2}+\frac{\tilde{y}-\tilde{x}}{2}$ $=16^\circ 15'+11^\circ 45'=28$	44. 28 *you see*; *from* the 2nd tear out, 4° 30′ *you see*. 28 *ta-mar i-na* 2-kam zi 4, 30 *ta-mar*
$\tilde{x}=\frac{\tilde{y}+\tilde{x}}{2}-\frac{\tilde{y}-\tilde{x}}{2}$ $=16^\circ 15'-11^\circ 45'=4^\circ 30'$	45. The igi *of* 3, the lengths, *detach*, 20′ *you see*. 20′ *to* 4° 30′ igi 3-*ti* uš *pu-ṭúr* 20 *ta-mar* 20 *a-na* 4, [30]
$X=3^{-1}\cdot\tilde{x}$ $=20'\cdot 4^\circ 30'$ $=1^\circ 30'$	46. {20′ *to* 4° 30′} *raise:* 1° 30′ *you see*. {20 *a-na* 4, 30} *i-ší-ma* 1, 30 *ta-mar*
$X=1^\circ 30'$ $\tilde{y}=28=21\cdot Y,\ Y?$	47. 1° 30′ the length, *that of* 2 the surface. [*What*]^a *to* 21, the widths, [*shall I pose*]^a 1, 30 uš *šà* 2 a-š[à *mi-na*] *a-na* 21 sag [*lu-uš-ku-un*]
$1^\circ 20'\cdot 21=28$ $Y=1^\circ 20'$	48. *which* 28 *give*[*s me*? 1° 20′ p]^a ose, 1° 20′ the width *šà* 28 *i-na-di*'[-*na* 1, 20 g]ar 1, 20 sag
$x=X-1=1^\circ 30'-1$ $=30'$	49. *that of* 2 the surface. *Turn back*. 1 *from* 1° 30′ tear out *šà* 2 a-šà *tu-úr* 1 *i-na* 1, [30 zi]
$y=Y-1=1^\circ 20'-1$ $=20'$	50. 30′ *you see*. 1 *from* 1° 20′ tear out, 30 *ta-mar* 1 *i-na* 1,20 z[i]
	51. 20′ *you see*. 20 *ta-*[*mar*]

[a] All these restitutions are mine. Restitutions in simple [ ] can be regarded as fairly well established, those in [( )] are reasoned guesses at a formulation, the factual contents of which can be relied upon.

[b] Line 6 quotes the value of the width in a way which would usually refer back to the statement, but which might of course refer to line 3; in any case, line 3 presupposes knowledge of the width, and line 5 refers to the length as a known quantity.

[c] BASE is a conjectural translation of the logogram KI.GUB.GUB (the testified Late Babylonian reading ki-du-du ~ *kidudûm*, "rites", makes no sense). GUB has two different Sumerian meanings, "to go" (readings du etc., cf. SLa § 268; used logographically for *alākum*) and "to stand, to erect" (gub, cf. SLa § 267; used logographically for *izuzzum* and *zaqāpum*). To judge from the logographic occurrences, the reduplication is used to indicate iterative and durative aspects.

ki can function as a virtual locativic verbal prefix, "on the ground" (cf. SLa, 306). A possible reading of KI.GUB.GUB is thus ki-gub-gub, "to stand/that which stands erected constantly on the ground".

[d] The transliteration in TMS writes 1. Still, the autography writes a sign after 1 which looks like 20 (and a damage to the tablet which has been read as an extra wedge). That is also the correct result, which is in fact used in line 8.

[e] The exact reconstructions of lines 14–16 are rather tentative, although the mathematical substance is fairly well-established thanks to the parallel of lines 28–31. It should be observed that even the extant signs until 1,20 *a* in line 14, and the š)]à and sa)]g of the following lines, are heavily damaged. The remaining traces may but need not correspond to my readings (according to autography and photo). The *aš-šum* of line 15 is needed, if not necessarily in that place, by the *šu-ta-ku-lu* of line 16, if I am right when reading it as the subjunctive mode of a stative (cf. lines 30f., and the subjunctive stative *qa-bu-ku* in line 6).

[f] The transliteration in TMS supposes that something is missing in the beginning of the line. The autography indicates that the line is simply written with indention.

[g] "Zu WA-ZU-*bi* im math. Susatect Nr. IX: Ich hatte mich für die Rezension von MDP 34 (= von Soden 1964 – JH) ziemlich gründlich damit beschäftigt und als mögliche Lesung *wu-ṣú-bi* als St. constr. eines sonst nicht bekannten *wuṣubbûm* notiert, diese Lesung aber dann als zu wenig gesichert nicht veröffentlicht." (Von Soden, private communication).

[h] "the various (things)" translates ḪI.A. This presupposed the assumption that the Sumerian suffix ḫi.a (designating a plurality of different entities) is used as a pseudo-Sumerogram in a nominal function (as a collective name for the collection of surface, length and width). It is also possible that ḫi-a stands as a pseudo-grammatical complement to a noun which was lost with the first part of the line.

TMS restitutes [. . .]-*ti* sag as *ša-la-aš-ti* sag and mistranslates the whole line as "]3 ⟨fois⟩ la longueur à 21 fois ⟨la largeur⟩ additionne" in order to get some apparent sense of the restitution. Apart from the mistake of "length" for "width" this mixes up "appending" and "accumulation". Only the first of these carries a "to" (*ana*) between the addends. A possible restitution which accepts the (somewhat dubious) *-ti* in the beginning of the line, which makes mathematical sense, which is as grammatically correct as can be expected in a text loaded with sumerograms, and which finally is in reasonable harmony with current usage, would be "17 (. . .?) and 4, of the four (*er-bet-ti*), widths, to 21, the ACCUMULATION" or ". . . to 21 ACCUMULATE". In lack of related passages I have, however, preferred to leave the question open.

[j] The transliteration in TMS renders the signs before *a-na* as ḪI.A. The A is in agreement with the autography, but the preceding sign looks very different from the ḪI of line 33. I have not been able to propose any better reading.

[k] The initial "10" is fully and the final *-mar* almost fully to be read on the autography, although they are left out in the transliteration. So, a repetition of the previous phrase appears to be the only possible restitution. Cf. also lines 45f.

[l] The lacuna consists of 1 or 2 signs, probably an epithet to the number 2' 6°. According to the autography, the first sign begins . This could belong to a TA, but such a restitution seems to make no sense. It could also belong to a TAG used logographically for *lapātum*, "to inscribe", and its derivations. This might make sense but would be without parallel ("2' 6° the inscribed").

The purely explanatory character of part A is revealed already in line 2, as the surface (which was never given) is referred to as known ("since . . .") (cf. also the restitution of the last part of line 1). Clearly, we are dealing with one equation in two (known) unknowns, uš = 30′, sag = 20′, and we are taught the way to transform it (in fact the same transformation as that of AO 8862 N^os 1–2: $xy + \alpha x \rightarrow xY$, $Y = y + \alpha$). In this way one can make sense of the "either . . . or . . . or" of lines 3–5 (Ú.UL . . . Ú.UL . . . Ú.UL), which governs three alternative ways to explain the transformation, but which has no place in an interpretation of the text as progressive argumentation (since the 1° 20′ created in line 3 is used in line 4, and line 5 repeats the contents of line 4), and which has therefore puzzled all commentators to the text.

If one follows the text step by step, it turns out that all of it can be read as an explanation of Fig. 15A, up to the end that explains that this is the point out from which problems containing such equations are to be solved, and finally sums up the main argument.

Part B deals with the same rectangle, but with a somewhat more complicated equation, $xy + x + y = 1$, and demonstrates how it is to be simplified "by the Ak-

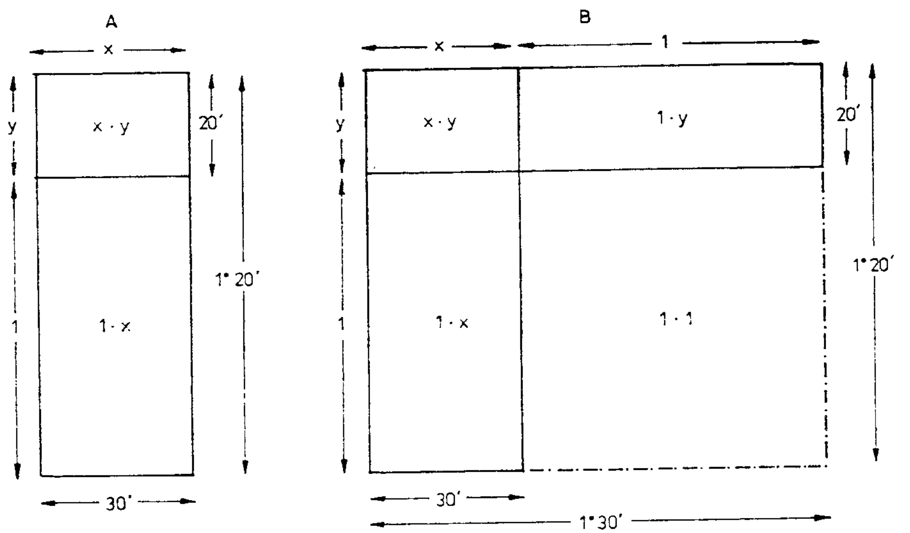


Figure 15. The geometrical configurations and operations described in TMS, parts A and B.

kadian (method)".[143] It can be followed on Fig. 15B. The method consists in completing the quasi-gnomon $xy+1 \cdot x+1 \cdot y$ into a rectangle XY, $X=x+1$, $Y=y+1$. X and Y are spoken of as "length" and "width" of "2 the surface" ($=XY$), in agreement with the figure.

Denominations of methods are rare in Mesopotamian mathematical texts, and one may wonder what makes the method of part B specifically "Akkadian". Which part of the procedure is it, furthermore, which deserves the label? My guess is that the term characterizes the quadratic completion in general, the basic trick needed to solve mixed second-degree equations. If anything, indeed, distinguishes the Old Babylonian "Akkadian" mathematical tradition from e.g. third millenium Sumerian mathematics, it will be its interest in second-degree algebra. Which more adequate name than the "Akkadian method" could then have been chosen for a trick which, simple as it may look once it is found, was perhaps the starting point for the whole fabulous development of "Akkadian" mathematics; a trick which, when it was first found, will certainly have been noticed as a novelty?[144]

It will be seen from line 14 that the values of both length and width are as-

[143] Truly, E. M. Bruins claims in the commentary in TMS (p. 67, and announced already pp. xi and 2) that the two parts deal with the same equation, and that part A expounds the master's own method and part B the alternative used by the Akkadians. For a number of reasons this is an impossible idea:

1) If the equation $xy+x=40'$ is to be equivalent with $xy+x+y=1$, one must presuppose $y=20'$. On the faith of line 6 E. M. Bruins claims (rightly, I suppose) that this value will have been given before (cf. my restituted line 1), from which he concludes that the text deals with a normal, complete set of two equations. Line 2, however, presupposes implicitly that the length is equally known ($10'$ the surface), while the value is stated explicitly in line 5 still without being calculated.

2) If the first half of line 10 were to be the result of a transformation belonging with the "Akkadian method", it could not precede the announcement of that method in the second half of the line.

3) In any case, the first half of line 10 is clearly in the style of statements; transformed equations are never restated in a similar form. Cf., e. g., the contrast with the formulation in lines 25f.

4) Finally, E. M. Bruins overlooks the identical statement in part C, as well as the fact that the procedure taught in part B is precisely the one used in part C.

It may be observed that the presumed "Susian" method is used in the Babylonian ("Akkadian") AO 8862 N^os^ 1–2, although N° 2 would have been greatly simplified had the "Akkadian method" been used.

[144] In this connection, the over-all character of Old Babylonian scribe school mathematics is worth reflecting upon. Greek mathematics, that other prototype of Ancient non-utilitarian mathematics, can be claimed to be essentially determined by its central problems (squaring the circle, doubling the cube, properties of conics, classification of irrationals, etc.). The great methodological innovations of Greek mathematics were made in order to solve (in a philosophically satisfactory manner!) these great problems. Old Babylonian scribal mathematics was, in as far as we concentrate upon its non-utilitarian aspect, determined by the methods at hand, and problems were chosen that would permit a brilliant display of the methods known to "the learned scribe", which makes scribe school mathematics a perfect parallel to other aspects of Old Babylonian scribal culture as presented, e. g., in the "examination texts". See my 1985a: 10–16 and passim, which discusses the difference between the two mathematical styles less coarsely than enforced by the limited space of a foot-note, and connects the different attitudes to their institutional and cultural context.

sumed to be known (though not given in the statement), and that they are used in the didactical exposition.

Part C contains a complete mathematical problem, a normal set of two equations in two unknown quantities "length" and "width". One of them is precisely the second-degree equation whose transformation was taught in part B, while the other (which can be transcribed $y+\frac{1}{17}(3x+4y)=30'$) is of the type whose transformation was explained in detail in TMS XVI (above, section VII.3). The values of length and width are still referred to during the solution (line 31), but only for identification, no longer as part of the argument. The identification must refer to something outside the written text[145], which can hardly be but a material representation more or less similar to Fig. 15B.

Lines 21 to 26, the transformation of the first-degree equation into $3x+21y=8°\ 30'$, must be presumed to follow the pattern from TMS XVI, and hence to be understood as an arithmetical transformation (we observe that the term for a coefficient, "as much as", recurs). Lines 28 to 33 appear to go by "naive geometry". For the next steps, lines 34 to 39, we are unfortunately not in possession of a didactical explanation. But some argumentation from Fig. 15B but similar to the accounting and scaling arithmetic of TMS XVI would at least be adequate, and is perhaps called for in line 27, which appears to connect to the following rather than the preceding section.[146] In any case, lines 39–44 solve the standard problem of a rectangle for which the area and the sum of length and width are known, the "false" length of which is $X=3\,(x+1)$, and the "false" width of which is $Y=21\,(y+1)$. The method is unfortunately not commented upon. Like the transformation of the linear equation the didactical explanation appears to have been given at an earlier stage, and the understanding now inherent in the vocabulary. Afterwards, the extended "real" length and width (those of "2 the surface") and finally the "real" length and width without extension are calculated (lines 45–51).

The whole tablet reflects a mathematics lesson. While part C represents a refined version of a standard problem known from elsewhere (VAT 8520, N^os^ 1–2, cf. note 146), parts A and B are didactical steps toward a particular aspect of the procedure needed to solve the complex standard problem. The other,

[145] The meticulous repetition of all steps appears to exclude a simple reference back to the known entities from section B.

[146] The argument can be imagined in the style of "false assumptions": If the length of the upper left rectangle in Fig. 13B is to represent 3 "true" lengths, the length of the upper right rectangle is 3 instead of 1. Similarly, if the upper left width represents 21 "true" widths, its extension will have to be 21 instead of 1. The sum of length and width of the total figure will then be $3+21=8°\ 30'$, cf. line 34. Furthermore, the total scaling factor for the area will be $21 \cdot 3 = 1`\ 3°$, and the area of the assumed surface will hence be $1'\ 3° \cdot 2 = 2`\ 6°$ (lines 36–39).

The last part of the interpretation seems to be confirmed by VAT 8520 N° 1 (MKT I 346f.). Here, an *igûm-igibûm* problem (translatable into $xy=1$, $x-{}^{6}/_{13}\,(x+y)=30'$) is solved in a similar way (extensions apart). The linear equation is transformed, it appears, into $7x-6y=8°\ 30'$, and a scaling factor of $7 \cdot 6 = 42$ is applied to "1 the surface". As the numbers 7 and 6 are to be retained by head, the transformation can be assumed to be performed mentally, not by means of any material representation beyond the changed conceptualization of the basic rectangle.

more general aspects of the procedure are supposed to be known from earlier lessons, and one of them was in fact explained in TMS XVI, as we have seen.

It has often been assumed that the Babylonian mathematical texts should be seen only as supplementary support for an oral tradition, and that the texts could only be understood by a person who knew beforehand what the whole thing was about.[147] The present investigation shows that the latter formulation is not as absolutely true as hitherto assumed, if only one knows the concrete meaning of the terminology. But still, the normal texts give the impression that they are a support for a teaching tradition making use of material representations outside the texts themselves, and referring to methods which had to be known beforehand. The material representations have still not been unearthed, and may be irretrievably lost (cf. above, chapter VI). The two Susa tablets, however, show us how the standard methods were taught, and the one just presented appears to refer more clearly perhaps than any other text to the naive-geometric representation.

IX. Summing up the evidence

The investigation has now arrived at a point where a summary of the results can reasonably be made. How far have we come in our understanding of the procedures, techniques and patterns of thought behind the Old Babylonian "algebraic" texts?

Chapters IV to VIII have by necessity been overloaded with details. If all conclusions were to be referred precisely to the single relevant pieces of evidence, the present chapter would make still heavier reading. As the conclusions to be drawn from the material have, however, been presented in scattered form all the way through, I hope that detailed references to the primary material can now be dispensed with.

On the negative side it will be remembered that the traditional arithmetico-algebraic interpretation left so many unexplainable points in the textual discourse that it can be safely dismissed (cf. most of the texts presented in chapter V). The possibility to make it work by minor corrections and ad hoc assumptions can also be disregarded, because no fundamentally arithmetical interpretation can map the structural distinctions within the vocabulary. Babylonian "algebra" was not a science about pure numbers and the ways in which they can be put into mutual relation, be it understood in analogy with Medieval rhetorical algebra as with F. Thureau-Dangin, O. Neugebauer and B. L. van der Waerden, or through that first-level criticism of the received interpretation which has been expressed by M. Mahoney.[148,149]

[147] This supplementary role is no distinctive characteristic of the mathematical texts. Similar claims could be made for most branches of Babylonian literature.

[148] Mahoney 1971.

[149] It should perhaps be emphasized once more that these remarks, as the whole of my investigation, regard the "algebraic" texts. They have no implications for those texts which are directly concerned with the properties of numbers, e.g. concerning inversion or continued multiplication; they, of course, cannot be denied the label "arithmetical".

Positively, the use of some sort of naive-geometric technique can be regarded as well-established. It fits all details of the textual discourse; it distinguishes operations which have to be distinguished according to the structure of the terminology; it agrees with the apparent metaphorical implications of many terms, including the puzzling *wāṣītum*, the "projection". The exact nature of the geometric representation is, however, open to doubt. We do not know to which extent the texts refer to a purely mental representation, though, truly, common pedagogical experience tells that mental geometry presupposes anterior intercourse with manifest geometry. We do not know the means (clay, dust, wax, or possibly sticks?) which were used to represent geometrical structures, relationships, and transformations manifestly, nor whether such representations should be thought of in analogy with modern geometrical drawings or as mere structural diagrams. These questions were discussed in further detail in chapter VI.

Apart from a two-dimensional extension of the "single false position", the naive-geometrical techniques were only used for problems involving a "surface", i.e. for problems of the second degree.[150] We can list these techniques as follows:

Firstly, there is the partition and rejoining of figures ("cut-and-paste"), which in ordinary "length-width" and "confrontation" problems is represented by the bisection and rearrangement of excessive or defective rectangles. In other, genuinely geometrical problems it is used more creatively[151], and as we shall mention in section X.4 there is evidence for continuity to later interests in the partition of figures.

Secondly, we have the completion technique, the supplementation of a gnomon or a quasi-gnomon into a square or a rectangle. This may be the technique which was spoken of as "the Akkadian (method)" in TMS IX.

Thirdly, we have the "scaling" technique, used e.g. when a non-normalized problem ($\alpha x^2+\beta x=\gamma$) is transformed into a normalized problem (in $z=\alpha x$), and to be understood perhaps as a change of measuring scale in one direction[152], perhaps as a proportional change of linear extensions in that direction.

The "accounting" technique may be claimed to have nothing specifically geometric about itself, and it was indeed set forth most clearly in the Susa text explaining the arithmetical transformations of a linear equation. Nonetheless, the counting of a specific entity (or the measurement of one entity in terms of another entity) is a necessary supplement to the specifically geometric techni-

[150] Inclusion of certain further texts would have forced us to modify this statement as well as the automatic identification of "surface"-problems with problems of the second degree. So, the "surface" problem Str. 367 (MKT I 259f.) is in reality of the first degree, but makes use of certain naive-geometric techniques all the same; other exceptions of various sorts could be mentioned. Already the first-degree "meadow" problems of VAT 8389 and 8391 could indeed be claimed to be exceptions; all of them are of the first degree, but formally they are of course concerned with surfaces, and part of the reasoning is made through imagined partition of a geometrical surface.

Problems "representing" prices, *igûm-igibûm* pairs etc. by dimensions of surfaces are not to be understood as exceptions but as "surface"-problems (cf. the use of the term "surface" in YBC 6967, above, section V.1).

[151] A very beautiful example is VAT 8512 (MKT I 341f.); see Gandz's deciphering of the procedure (1948: 36), or the more detailed analysis of the text in my 1985: 105.15ff.

[152] This would hardly bother the Babylonians, who appear to treat a rectangle of length 45 nindan and height 45 cubit as they treat "any other" square (see my 1985: 53—63).

ques, without which no "analysis" by means of geometry (be it naive or based on Euclidean demonstrations) can reproduce the results of arithmetico-rhetorical algebra. The "accounting" and "scaling" techniques are of course closely related.

Hardly to be counted as regular "techniques" but still parts of Old Babylonian naive-geometric methodology are the reasoning by various "false" assumptions and the ability to take any adequate entity of a geometric configuration as that "basic" entity which is to be submitted to the habitual standard operations.

The global picture arising from the use of these techniques and quasi-techniques is the predominance of constructive procedures; only a single pre-established, fixed geometrical standard configuration–the one presented in Fig. 2, and visible as a basic grid in Fig. 14 A–has suggested itself during the investigation.

The investigation was only peripherally concerned with first-degree techniques. Even on the basis of the restricted material presented here can it be seen, however, that most reasoning about first-degree problems is verbal and basically arithmetical in character. Like second-degree problems, however, problems of the first degree are dealt with by means of "accounting" and various "false" assumptions. Like the second-degree "algebra" the reasoning on questions of the first degree is also concrete, bound to representations of manifest entities (mental representations in most cases, I guess). Hence of course the predilection for "false assumptions", which consist precisely in taking one entity, real or imagined, as a representative for another, normally unknown quantity.

It was recognized already in the early 1930es that Babylonian "algebra" problems were constructed from known solutions. In the case of the "series texts", where often large numbers of problems deal with the same figure it is also obvious that the user of the texts would know the solution beforehand. The didactical Susa texts have now shown us (as it was also apparent from the tabulation in AO 8862 N° 1) that even the student would, at least in certain cases, have been told the solution beforehand, which would permit an identification of the entities involved in the procedure and also an explanation of the way it works.

The backward construction has traditionally been taken as evidence that the aim of the mathematical texts was the teaching of procedures and techniques.[153] The insights gained from the improved understanding of the vocabulary, regarding the use of naive-geometric justifications, and from the didactical Susa texts show us that the aim was not only technical know-how but also understanding, "know-why". This helps us grasp how Babylonian mathematics was at all possible at its actual level. If its sole social justification had been a teaching enterprise dominated by empty rote learning, from where should it then have got the necessary intellectual inspiration and surplus?

A summary of the results concerning the details of terminology would mainly become a repetition of chapter IV, which was in fact an anticipation of the results established in later chapters. I shall therefore only refer to Table 1 as the briefest possible summary of terminological details. On the general level, however,

[153] Since our texts are school-texts and not practitioners' notebooks this may seem their only possible aim. The occurrence of problems of the third degree for which the Babylonians knew no general solution, and which are therefore treated by non-generalizable tricks, show that another aim was possible and in fact also present at least occasionally: That of demonstrating the mock ability of the teacher. Cf. also above, note 144.

the somewhat floating character of the terminology should be remembered. Only as a first approximation can it be called "technical". It appears not to have been stripped completely of the connotations of everyday language, nor does it possess that stiffness which distinguishes a real technical terminology. We should rather comprehend the discourse of the mathematical texts as a highly standardized description in everyday language of standardized problem situations and procedures, and we should notice that the discourse is never more, but sometimes less standardized than the situation described.[154] As everyday life contained no second-degree problems (be it the life of a professional scribal surveyor or accountant), terms taken from everyday language would of course have to be applied differently when describing procedures of second-degree "algebra" than in other texts. In as far as the use in such other texts is taken to represent the "basic meaning", the terms of the "algebra" texts will appear in the quality of standardized metaphors,–whence that impression of a technical terminology which is conveyed by standard problems.

The Sumerographic writings inside the otherwise Akkadian mathematical texts presents us with a special interpretative problem. Are they not to be interpreted as technical terminology?

In order to answer this question we have to distinguish different sorts of Sumerographic writing. On the one hand we have a restricted number of terms which are invariably written in Sumerian: uš, sag, a-šà, igi, íb-si$_8$, ba-si$_8$. Even inside this group there is a certain variability, ba-si$_8$ and igi giving rise to Akkadian loanwords and hence spoken with certainty as Sumerian words, and a-šà being often provided with phonetic complements and hence probably spoken in Akkadian. None the less, these terms can be regarded as technical and free of everyday connotations, as it is made especially clear when uš and sag used outside the basic representation are suddenly replaced by corresponding Akkadian words (cf. note 75).

Then we have the large number of pseudo-Sumerian writings, where Sumerograms are used logographically. In as far as the logographic meanings of these Sumerograms are not specifically reserved for mathematical texts they are no more and no less technical than the Akkadian words which they replace, or, alternatively, they are technical with respect to the scribal craft but not with regard to mathematics.

Finally we have a domain of indeterminate extension, that of Sumerograms used as possible alternatives for Akkadian writing but used ideographically. We have met one indubitable instance, viz. zi quoted in Akkadian as an infinitive in TMS XVI, which proves that the category is not empty. But this was an exceptional case, and other instances may be impossible to disclose. Especially the very compact and very ungrammatical Sumerographic writing of the series

[154] Seen in a long-run perspective this is of course also true of modern mathematical terminology. New theoretical developments give rise to new applications of old terms. Just think of a creature like the "infinite-dimensional vector space", in which at most "infinity" can still claim a classical value. Since the time when mathematical terms were given precise definitions, however, every extension by analogy and metaphor constitutes a clear and definite break. This was apparently different in Babylonian mathematics, which saw no absolute conceptual border-line between standard-situation and analogous extension.

texts (ungrammatical both from an Akkadian and from a Sumerian point of view) may be suspected to belong here.

The remainder of the present chapter shall deal with two questions of more general character: The relations of our Old Babylonian discipline to the categories of later mathematical thought, and its relation to the intellectual style of its own age.

Throughout this chapter I have spoken of Old Babylonian "algebra", not algebra. But was Babylonian "algebra" an algebra? Put in this form the question will of course have to be answered by a definition, which is not in itself a very fruitful way. We shall learn more by asking, in which respects Babylonian "algebra" was similar to Medieval or post-Renaissance algebra?

We should start from the outside, observing the uses to which the Babylonian discipline was put–and not put. In later times, algebraic techniques have been used to find the solution to problems which could not be solved by direct computation. We have no Babylonian texts which suggest such uses of the naive-geometric "algebra". On the contrary, the specious problems which had to be constructed in order to give occasion for the display of "algebraic" second-degree techniques suggest that no real uses were known. The abundance of realistic manpower- and brick-problems demonstrate that the Babylonian schoolmasters did nothing to hide a possible real-life importance of their teaching. "Algebra" never served to find a numerical value unknown in advance. In that respect its function was very different from that of algebra.

Recognition of this difference should not force us into the opposite extreme, and should not make us believe that naive-geometric "algebra" was nothing but an investigation of certain numerical properties of squares and rectangles, a peculiar sort of geometry. In chapter I I introduced the concept of a "basic conceptualization". The uš and sag are indeed basic in the sense that they are used to represent other quantities, the arithmetical relations between which can be mapped by the relations between the lengths and widths of rectangles. In YBC 6967 we have seen how a pair of numbers with known product and difference was represented by the dimensions of a rectangle, made visible in the text by the explicit reference to a "surface". Other texts would show a wide variety of quantities being represented as linear quantities, more or less explicitly mentioned. Especially interesting are certain cases where the text appears to distinguish between the linear extensions of a real figure, supposed, we may guess, to be situated in the terrain, and the corresponding extensions of a representing figure (drawn perhaps in the dusty schoolyard), even though the two coincide numerically.[155] Naive-geometric analysis of quadrangles is hence used as a means to

[155] This is the most probable implication of the distinction between "length" and "true length" in TMS XVI (section VII.3). In BM 13901 № 14, the "confrontation" spoken of in the statement and that inherent in the procedure can also be seen to be kept apart through the multiplication by 1 in rev. I 9 (section VIII.1). Finally, TMS XIX appears to designate a "representing length" 1 as the "counterpart" of the "real length" 1 (cf. below, note 176).

Abstract distinction between a mentally conceived "real entity" and an equally mental "representing entity" may be too abstract to be expected in a Babylonian context. A reasonable guess would be that the traces of an explicitly distinguished representation are also traces of a concrete, material representation.

solve problems from other domains, be they artificial and the solutions known beforehand to exist as regular numbers. Though "algebra" was in all probability not used instrumentally in nonartificial situations, it was obviously taught as a virtual instrument.[156]

In virtual use and scope, "algebra" was hence related to real algebra. Can a similar claim be made for its "essence", its internal structure and characteristics? In a criticism of the unreflected use of the modern term to characterize a Babylonian discipline M. Mahoney has listed three characteristic features of developed algebra[157]: Firstly, the employment of "a symbolism for the purpose of abstracting the structure of a mathematical problem from its non-essential content"; secondly, the search for "the relationships (usually combinatory operations) that characterize or define that structure or link it to other structures"; thirdly, abstractness and absence of all "ontological commitments".

Taken at the letter, and allowing only for divergence "by degree rather than kind", these features are only valid and only meant to be valid for post-Vietan algebra understood as a scientific discipline. Already Medieval or more recent practitioners' algebraic calculation will only deserve the label "algebraic approach". In the same strict language, Old Babylonian "algebra" is algebraic "in approach": It cannot be claimed to possess a real symbolism. Still, even if the uš and sag are no more symbols than the Diophantine ἀριθμός or the Medieval thing, their use as ingredients of a "basic representation" serves precisely if only implicitly "the purpose of abstracting the structure of a mathematical problem from its non-essential content". Secondly, a number of systematic texts (especially among the series texts, but even BM 13901 can be mentioned) are in fact systematic investigations of the relationship characterizing the uš-sag-structure. Only the third criterion is not fulfilled even tendentially–unless we will claim that the use of a common basic representation is already virtual abstraction.

The "essence" of algebra can also be approached in another way, which links the beginnings of scientific algebra more clearly to the Medieval Art of Algebra and to the practitioners' algebra of the Modern era. In his "Introduction to the Analytic Art", in which Vieta aimed at bringing to light the hidden gold of algebra and almuchabala, he found the true essence of that art in the Ancient Method of Analysis, "assuming that which is sought for as if it were admitted [and working] through the consequences [of that assumption] to what is admittedly true".[158] This is exactly what we teach school children to do when solving an equation: "You treat x precisely as if it were an ordinary number". Apart from the known values used for identification purposes during explanations, but

[156] There is no reason to be overly astonished or scandalized on behalf of the poor scribe school students on this account. Apart from a modest (not to say infinitesimal) minority of the school children who have been taught second-degree algebra during the latest $3^1/_2$ millennia, their situation has been exactly the same, when not worse. Unless you make interpolation in trigonometrical or similar tables, physics at least at the level of Galilean ballistics, or something similar, second-degree algebra can only be used to train second-degree algebra.

[157] Mahoney 1971: 372.

[158] Chapter 1, ed. Hofmann 1970: 7; I follow Witmer's translation (1983: 11). Vieta cites Theon's definition of analysis. The gold metaphor is found in the dedicatory letter (ed. Hoffman 1970: xi).

not as steps in the mathematical argument (cf. TMS IX, part C), it is also a precise description of the Old Babylonian procedures. In this respect, too, Old Babylonian "algebra" is therefore algebraic, or at least characterizable as "naive-geometric analysis".[159]

Was "algebra" then an algebra? If we apply M. Mahoney's criteria, it was not. Babylonian mathematics differed more than in degree from the discipline founded by Vieta and continuing through Descartes and Noether. But it was "algebraic in approach", belonging in full right to any family which is able to encompass both al-Khwārizmī, Cardano and Noether. Anybody using confidently the expression "Medieval algebra" can with equal confidence speak of "Babylonian algebra".

Instead of relating our subject to categories of later times we may compare it to the general cognitive style of its own time, thereby regarding it as one aspect of the thought of its times, on an equal footing with others.

In their introduction to a famous "essay on speculative thought in the Ancient Near East"[160], H. and H. A. Frankfort characterize it as "mythopoeic". There are several facets to the concept, but its main implication is that the phenomenal world is no object, no "it": it is a "thou", an animated individual. In as far as this is an adequate description it excludes a scientific cosmology in the modern sense, a cosmology extrapolated under theoretical guidance from rational experimentation and hence in the final instance from technological practice. (I agree with any critical mind who finds this description short-circuited.) In this sense, it is true, we find no scientific cosmology in Ancient Mesopotamia. In the same sense it is indeed difficult to connect a scientific cosmology to any poetical or religious world-view, and so far it is therefore not obvious that the domination of cosmology by myth should imply that Ancient Mesopotamian thought in general be mythopoeic.[161]

Now, not everything in Babylonian thought was speculative; much of it was founded on social practice[162] or on technological practice. In both of these, and especially in the latter, the object-aspect of the external world, which under this view is not just "phenomenal", must be expected to impose itself. It is therefore not astonishing that it seems "difficult to accept [mythopoeiecy] as an adequate characterization" of "the intellectual adventure of ancient man" as "documented in the corpus of administrative, commercial, technical and other genres".[163]

[159] We observe that even the argument by a single false position is a primitive sort of analysis albeit arithmetical. Take e.g. the problem that a "heap" and its fourth is 15. For lack of an x permitting us to rewrite the 15 as $1^1/_4x$ one takes the number to be known, viz. as 4, etc.

[160] H. Frankfort et al. 1946: 3–27.

[161] Precisely this question is raised regarding Babylonian mathematical thought by Mahoney (1971: 370).

[162] That even large parts of mythology were founded on social practice has been argued by Jacobsen (1976; and already in H. Frankfort et al. 1946: 125–219). A proverb like "Workmen without a foreman are waters without a canal inspector" demonstrates clearly that Babylonian overseer-scribes were as able to see their fellow beings under the aspect of objects as their myths were to see nature as a fellow being (H. Frankfort et al. 1946: 203).

[163] Larsen 1987: 205.

Our algebraic texts constitute another exception to the presumed mythopoeic rule. Truly, AO 8862 carries an invocation of the scribal goddess Nisaba on its edge; but this and other similar inscriptions are totally isolated from the rest of the text, which treats its subject not as a "thou" having the "unprecedented, unparalleled, and unpredicatable character of an individual, a presence known only in so far as it reveals itself"[164], but as a fully predictable, manipulable and comprehensible object. No wonder, since Babylonian algebra was definitely not "speculative", i.e "regarding", but active, technical construction. According to the Frankforts' dichotomy it is "modern", dealing with lenghts, widths and surfaces and with its problem-situations as "objects and events [. . .] ruled by universal laws which make their behavior under given circumstances predictable", and which "can always be scientifically related to other objects and appear as part of a group or a series".[165]

This does not mean that Babylonian mathematics and technical thought in general was modern, only that its difference from modernity cannot be grasped by the Frankfort dichotomy. Nor should the secular rationality of Hammurapi's "Code" make us mistake this collection of concrete decisions for an abstract, general law-book in the style of Roman law.[166] A recent investigation of the cognitive character of Babylonian divination science[167] tries to get beyond such mistakes through reference to C. Lévi-Strauss's distinction between "hot" and "cold" societies, between the "savage" and the "domesticated" mind, between "the science of the concrete" and that of "abstract thought", illustrated by the distinction between the "bricoleur" (a cross-breed between the "tinkerer" and the "Jack of all trades") and the engineer.[168]

In the Lévi-Strauss illustration, engineering technology is thought of as developing specialized tools for the job to be done. The bricoleur, on the contrary, takes what happens to be at hand and fits it together as best can be done. "Domesticated" science and thought is seen analogously as building on abstract concepts; the "savage mind", on the other hand, classifies the categories and oppositions of e.g. their social world using pre-existent entities as classifiers and analogies.[169] While concepts are "wholly transparent with respect to reality", meaning n o t h i n g but their conceptual content, a pre-existent concrete entity used as a symbolizer is a s i g n, preserving to some extent the cultural meaning it possesses in itself and imparting it to those other entities for which it is used as a classifier.[170] (Being a member of the "Arrow Clan" may imply swiftness!)

In his investigation of the Babylonian lexical lists and omen literature, M. T. Larsen comes to the conclusion that many features (the search for classifying

[164] H. Frankfort–H. A. Frankfort, in: H. Frankfort et al. 1946: 5.

[165] Ibid.

[166] See Renger 1976: 229 and passim. The validity of the description is not affected by the discussions whether the decisions were considered paradigmatic or not.

[167] Larsen 1987.

[168] Lévi-Strauss 1972: 16ff.

[169] The opposition between day and night can thus be used as an analogy or "model" for the two moieties of a tribe; clans labeled after animals are part of common lore, not signifying, however, that the clan members assume descent from the real animal in question, but affinity in some higher sense (cf. ibid. 142f. and 149).

[170] Ibid. 20

order and the postulate of direct causation, partly built on recorded experience and partly on analogic thought) can be described as "savage". Other features of the omen literature are, from its Old Babylonian beginnings, better described as "semidomesticated": The intent to engineer the future, the attempt to make exhaustive listings of all possible omina (which presupposes writing, a main domesticator) and the way in which lacunae in the empirical record are filled out by means of abstract, logical rules–rules which are in fact formulated explicitly in a Neo Assyrian compendium. All in all, however, the global logic of the divination prevented the apparent steps toward "domesticated science" from leading to any ultimate breakthrough.

How are we then to regard Old Babylonian mathematics? Is it also "lukewarm", blocked midway between a neolithic "cold" society and our modern "hot" world?

Several features, at least, look "savage". It was claimed time and again in the preceding chapters that a pattern of thought was "concrete", which sounds very much like the classification by means of pre-existent, concrete entities used as signs. But let us look at the "concrete" argument in VAT 8389 No 1. In this case "concreteness" means that the mathematical structure is thought in terms of the real entities involved. There is no distinct, concrete signifier, no sign imparting to the "meadows" any characteristics beyond those of possessing an area and to yield a specified rent per area unit. "Concreteness" simply means "absence of any explicit abstract signifier or abstract calculating scheme" (no x or ἀριθμός, no standardized "double false position").

In second-degree problems like those of BM 13901 or AO 8862 (the "basic representation" itself) we see the same sort of concreteness. "Naive geometry" consists precisely in taking geometrical entities at their phenomenal face value, without submitting them to theoretical reflection through which their properties and mutual relationships might be formulated as abstract principles.[171]

In cases where something else is dealt with by means of a mapping on the basic representation, be it the number pairs of a table of reciprocals, prices, or real linear extensions, we seem to come closer to the use of concrete entities as signs. Even here, however, we should take care. There is no hint that a price represented through a length has anything in common with that line, except, precisely, the relevant characteristic, the measuring number. No text whatever suggests anything similar to the swiftness of the Arrow Clan. On the contrary, the representation is normally only visible through the designations of the operations performed ("breaking", "making span", etc.). Only ocasionally do we find a "surface" or a "true length", etc. In its function, the basic representation can be regarded as an abstract instrument.

[171] The definitions, axioms and postulates of the *Elements* are precisely such a set of abstracted principles, and the deductive build-up of the whole work constitutes a conscious attempt to build the complete argument on these. Truly, the abstracted system is not complete, as it is well known, and at times "naive" knowledge is made use of implicitly; and conversely it is obvious that Old Babylonian "naive" geometry is full of implicit abstraction: assumptions on the calculability of areas as products, knowledge of arithmetical rules, etc. Neither observation affects the fact that we have to do with fundamentally different projects.

Places where the description of "savage thought" is really relevant for Old Babylonian algebra are its terminology, and hence its operations. Like Lévi-Strauss's "concepts", technical terms are "wholly transparent", meaning nothing but their direct technical implication. They have no connotations. Like his "signs", descriptive metaphors, even when used in a standardized way as long as the situation itself is standard, carry a load of everyday connotations, causing e.g. its users to "tear out" rather than "break off" a square from another square. The terminology being only partly technicalized, we might characterize it as "semi-savage".

A second "semi-savage" aspect of Old Babylonian algebraic mathematics is constituted by the series texts. As I have not dealt with them above, I shall only state briefly that the listings of large numbers of variations on the same type of equation is a parallel to the way all possible liver shapes are listed in the omen lists, and to the lexical lists. But it is no perfect parallel. While the lists are first of all additive and aggregative listings, introducing hierarchical ordering only in so far as this reflects "the surrounding highly stratified society"[172], the series texts are constructed in main sections, first order subdivisions, and cartesian products of second-order subdivisions.[173]

In the case of the omen text, the Neo-Assyrian compendium formulating explicit, abstract rules was an unprecedented innovation, at least as far as the written record has been excavated. In mathematics, the corresponding step can be demonstrated to have been taken already by the late old Babylonian period, viz. on the Susa text TMS XVI, which furthermore looks very much as a written documentation of a sort of didactical explanation which would normally be given orally. Didactical explanation does not in itself constitute theoretical reflection on abstract principles, and it was thus no step leading automatically to abstract, deductive mathematics. But it was a starting point from which a critically inquisitive intellectual environment might have been able to proceed indefinitely long. Sticking to the cold-hot metaphor we may say that Old Babylonian algebra was after all not only "lukewarm" but also inflammable. Further development of the discipline was not blocked by any immanent intellectual structure reflecting the over-all social and intellectual climate, as was the case of divination science. The blocking factors resided directly in global social and intellectual conditions: The scribal school was only moderately inquisitive and definitely not critical; the prime reason for interest in mathematical knowledge beyond the requirements of direct utility was professional pride and social prestige rather than curiosity and openness to the infinite possibilities of an unknown world. Furthermore: By the end of the Old Babylonian era, the scribal environment changed socially and intellectually, cutting off even the supplies for that sort of mathematical research which had been undertaken until then.[174]

[172] Larsen 1987: 211.

[173] The system is clearly visible in the symbolic transcriptions of three sections of VAT 7537 in MKT I 474f.

[174] For the motivations of Old Babylonian non-utilitarian mathematical activities, cf. above, note 144. The changes after the end of the Old Babylonian era are discussed in my 1980: 28f.

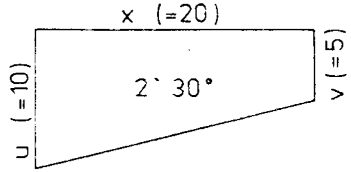
X. *The legacy*

So, after the end of the Old Babylonian era, second-degree algebra vanishes from the documentary horizon for many centuries–as do in fact all specific traces of mathematics teaching. That does not mean, however, that Old Babylonian mathematics was a complete mathematical dead-end without consequences for later mathematical cultures. On the contrary: though rarefied for a millennium below the level of archaeological visibility, the Old Babylonian tradition was to excert its influence on several of the sources of Modern mathematics.

Before looking directly at the evidence for such influence we shall, however, investigate yet another Old Babylonian text, one in which the conceptual dynamics of Old Babylonian algebra can be glimpsed.

X.1. A possible shift in the conceptualization: IM 52301 N° 2 (Baqir 1950a, improved transliteration in Gundlach–von Soden 1963: 252f.)

The text in question is problem N° 2 from IM 52301, perhaps the youngest of the (northern) Tell Ḫarmal mathematical tablets. It deals with a real geometric trapezium[175], and reduces the problem to one of "surface and confrontations equal to number". Besides being a beautiful specimen of "representation", the text is interesting because of its deviations from normal usage, which suggest a tendency toward changing or looser conceptualizations. It runs as follows (the marginal drawing is not in the tablet):



Obverse

16. *If to two-third of the accumulation of the upper* width
šum-ma a-na ši-ni-ip ku-mu-ri sag *e-li-tim*

$^2/_3 \cdot (u-v)+10=x(=20)$

17. *and* the *lower*, 10, *to my hand*[a] I have appended: 20 the length *I have built*. The width
ù ša-ap-li-tim 10 *a-na qa-ti-ia* daḫ-*ma* 20 uš *ab-ni* sag

$u-v=5$

18. {. . .}[b] the *upper*, *over* the *tower* 5 *goes beyond*.
{*e-li*} *e-li-tum e-li ša-ap-li-tim* 5 *i-te-er*

$\frac{u+v}{2} \cdot x = 2` 30°$

19. The surface is 2` 30°. *What my* lengths? You, by your saying[b], 5 *which it goes beyond*
a-šà 2, 30 *mi-nu-um* uš-*ia* za-e TUK-zú-dè 5 *ša e-te-ru*

Putting $u+v=Z$:
$x = ^2/_3 \cdot Z+10$
$(Z/2)\cdot(^2/_3 \cdot Z+10)$
$=2` 30°$
or, with an adequate choice for α:

20. 10 *which you have appended*; 40′ *of* the *two-third*, *my factors of both*(?)[c]; *inscribe:*
10 *ša tu-iṣ-bu* 40 *ši-ni-pé-tim a-ra-ma-ni-a-ti-a lu-pu-ut-ma*

[175] From the mathematical structure alone, Bruins' interpretation (1966: 207ff.), viz. a triangle cut by a transversal, cannot be excluded. But the expression "upper length" in rev. 17 speaks definitely against it, as does the absence of partial areas from the statement.

$(Z/2)\cdot(Z+2\alpha)$
$=(^2/_3)^{-1}\cdot 2` 30° = 3` 45°$

21. The igi of 40′ *of* the *two-third detach:* 1° 30′ *you see.* 1° 30′ {. . .
i-gi 40 *ši-ni-pé-tim pu-ṭú-ur-ma* 1, 30 *ta-mar* 1, 30 {*ḫi-pí*(?)-*ma*

22. . . .}[b] *to* 2` 30°, the surface, *raise:* 3` 45° *you see.*
4⌈5 *t*⌉*a-mar* 45} *a-na* 2, 30 a-šà *i-ši-ma* 3, 45 *ta-mar*

$Z\cdot(Z+2\alpha)=7` 30°$

23. 3` 45° *repeat:* 7` 30° *you see.* 7` 30° *your head*
3, 45 *e-ṣí-ma* 7, 30 *ta-mar* 7, 30 *ri-iš-ka*

$\alpha=\{(^2/_3)^{-1}\cdot {^1/_2}\}\cdot 10$
$=45'\cdot 10 = 7° 30'$

24. *may retain. Turn back.* The igi of 40′ *of* the *two-third detach*
li-ki-il tu-ur-ma i-gi 40 *ši-ni-pé-tim pu-ṭú-ur*

Reverse

1. 1° 30′ *you see.* 1° 30′ *break:* 45′ *you see; to* 10 *which you have appended*
1, 30 *ta-mar* 1, 30 *ḫi-pí-ma* 45 *ta-mar a-na* 10 *ša tu-iṣ-bu*

2–4. *raise:* 7° 30′ *you see* {. . .}[b]
i-ši-ma 7,30 *ta-mar* {7, 30 *ri-iš-ka li-ki-il tu-ur-ma* i-gi 40 *pu-ṭú-ur-ma* 1, 30 *ta-mar* 1, 40 *ḫi-pí-ma* 45 *ta-mar a-na* 10 *ša tu-iṣ-bu i-ši-ma* 7, 30 *ta-mar*}

$Z^2+2\cdot 7° 30'\cdot Z = 7` 30°$

5. 7° 30′ the *counter* {. . .}[b] *part lay down: Make span:*
7, 30 *me-eḫ*-{*ša*}-*ra-am i-di-ma šu-ta-ku-il-ma*

$(Z+7° 30')^2$
$=7` 30° + 56° 15'$
$=8` 26° 15'$

6. 56° 15′ *you see.* 56° 15′ *to* 7` 30° *which your head*
56, 15 *ta-mar* 56, 15 *a-na* 7, 30 *ša-ri-iš-ka*

7. *retains append:* 8` 26° 15′ *you see.* The equilateral[d]
ú-ka-lu ṣí-ib-ma 8, 26, 15 *ta-mar* ba-se-e

$Z+7° 30' = \sqrt{8` 26° 15'}$
$=22° 30'$

8. of 8` 26° 15′ *make come up:* 22° 30′ *its equilateral*[d]; *from* 22° 30′
8, 26, 15 *šu-li-ma* 22, 30 *ba-su-šu i-na* 22, 30

$Z=22° 30' - 7° 30' = 15$

9. the equilateral[d] 7° 30′, *your takīltum, cut off,*
ba-se-e 7, 30 *ta-ki-il-ta-ka ḫu-ru-uṣ*$_4$

$\frac{u+v}{2}=Z/2=7° 30'$

10. 15 the *left-over*. 15 *break:* 7° 30′ *you see,* 7° 30′ the *counterpart lay down:*
15 *ši-ta-tum* 15 *ḫi-pí-ma* 7, 30 *ta-mar* 7, 30 *me-eḫ-ra-am i-di-ma*

$\frac{u-v}{2}=5/2=2° 30'$

11. 5 *which* width *over* width *goes beyond break:*
5 *ša* sag *e-li* sag *i-te-ru ḫi-pí-ma*

$u=\frac{u+v}{2}-\frac{u-v}{2}$
$=7° 30' + 2° 30' = 10$

12. 2° 30′ *you see.* 2° 30′ *to* the *first* 7° 30′ *append:*
2, 30 *ta-mar* 2, 30 *a-na* 7, 30 *iš-ti-in ṣí-im-ma*

$v=\frac{u+v}{2}+\frac{u-v}{2}$
$=7° 30' - 2° 30' = 5$

13. 10 *you see; from* the *second* 7° 30′ *cut off.*
10 *ta-mar i-na* 7, 30 *ša-ni-im ḫu-ru-uṣ*$_4$

	14. 10 the *upper* width; 5 the *lower* width. 10 sag *e-li-tum* 5 sag *ša-ap-li-tum*
Proof: $u+v=10+5=15$	15. *Turn back:* 10 *and* 5 *accumulate,* 15 *you see.* *tu-ur-ma* 10 *ù* 5 *ku-mu-ur* 15 *ta-mar*
$^2/_3 \cdot (u+v)=10$	16. The *two-third of* 15 *take:* 10 *you see, and* 10 *append:* *ši-ni-ip-pé-at* 15 *le-qé-ma* 10 *ta-mar ù* 10 *ṣi-ib-ma*
$x=10+10=20$ $\frac{u+v}{2}=7°\ 30'$	17. 20 *your upper* length. 15 *break:* 7° 30′ *you see.* 20 uš-*ka e-lu-um* 15 *ḫi-pí-ma* 7, 30 *ta-mar*
$\frac{u+v}{2}\cdot x=2'\ 30°$	18. 7° 30′ *to* 20 *raise:* 2′ 30°, the surface, *you see.* 7, 30 *a-na* 20 *i-ši-ma* 2, 30 a-šà *ta-mar*
	19. *So* the *having-been-made.* *ki-a-am ne-pé-šum*

[a] I.e. a number 10 which is "at my disposition" without being defined in relation to the figure.

[b] The text contains a number of repetitions, other erroneous insertions etc. due to faulty copying. Those of obv. 18 and rev. 5 were already pointed out by T. Baqir. Those of obv. 21f. and rev. 2–4 (the first of which has been induced by the phrase, 1,30 *ta-mar* 1,30 common to obv. 21 and rev. 1, while the second is provoked by the 7,30 *ta-mar* common to obv. 23 and rev. 2) follow from analysis of the procedure.

The reading of zú as a homophonic mistake for zu in obv. 19 was given in von Soden (1952a: 49). That of TUK as dug_4 was suggested by Baqir (1950a: 146).

[c] "factors of both" is a tentative translation of *aramaniātum*, a plural form known from nowhere else. The term is an epithet to 40′, which multiplies the sum of the widths. The term thus appears to suggest two (identical) factors multiplying the members of a sum. In agreement with this, von Soden (1952a: 50) suggests conjecturally the word to be a loanword from Sumerian ara-man, "times"-"two", i.e. "factors of both".

[d] The "equilateral" of rev. 7–9 is written in syllabic writing. In rev. 7 and 9, the form is BA.SE.E, indicating that the form normally written ba-si_8 (which alternates with íb-si_8) was pronounced in Sumerian. (In a similar fashion, the text writes a syllabic i-gi instead of the normal igi.) In rev. 8, the form is a nominative with suffix, *ba-su-šu*, suggesting an Akkadianized form *basûm*. The accusative form in rev. 7 could in principle be a construct state of the same form, but the genitive in rev. 9 cannot, since the rest of the text is written with full mimation. It must render a genuine Sumerian pronunciation of the term.

Both forms confirm, as does the homophonic shift from si_8 to si in certain texts, that the term was not read as a logogram for an Akkadian word (*mitḫartum* being the normal assumption), at least not when used for the extraction of a square-root.

In AO 17264 (late Old Babylonian or early Kassite) the forms *ba-si-e-šu* and *ba-si-šu* are found (MKT I, 127). Even here, the equilateral is "asked for" (*šâlum*).

Before drawing any conclusions from the way the text formulates its subject-matter we should of course make sure that this subject-matter is understood correctly. Is the interpretation in the marginal commentary adequate, apart from the anachronism inherent in the use of modern algebraic symbolism? Should we not instead expect that the problem was seen as one in two unknowns (a "length-width"-problem) the product and difference of which are known (Z and $Z+2\alpha$, in the symbolism of the margin)? Or, if it is to be understood in terms of one unknown ("surface and confrontations"), is the average width $\left(\frac{u+v}{2}=Z/2\right)$ not the entity which would normally be chosen by a Babylonian?

Both answers should probably be answered by "yes"; we should perhaps expect the problem to be comprehended in two unknowns, and if not, the average rather than the aggregated width would be a normal Babylonian unknown. But in the first case we would also expect that the difference between the two be really calculated; instead, the scaling factor 1° 30′ is bisected before the multiplication is performed, without any other reason calling for that sequence of operations. In the second case, the operation in obv. 23 would have been a "raising", the normal scaling multiplication (cf. section V.5, BM 13901 No 3), and that of rev. 10 would have been a reverse scaling. Instead, the first is a "repetition" and the second a "breaking", concrete operations which indicate that operations belonging with the standard procedure are only found from obv. 24 to rev. 9, and thus that the sum of the widths, i.e. the *15* found in rev. 9, is the quantity looked for in that procedure. All normal Babylonian habits notwithstanding, the marginal commentary appears to map the original procedure.

If we look at the formulation of the text, it is obviously close to the style known from Old Babylonian algebra in general, so much so, in fact, that only lack of feeling for the stylistic implications of the naive-geometric procedures (most notably the identification of the 7,30 of rev. 9 as a *takīltum*, i.e. as the same as that of rev. 5) has prevented earlier investigators of the text from identifying correctly the dittographies of obv. 21f. and rev. 2–4.

Apart from the erroneous repetitions (which are obviously due to copying errors and which therefore presuppose the existence of a more correct original) and the syllabic writings of Sumerian terms there are, however, certain deviations from normal usage which can hardly be explained unless we assume some slackening of normal conceptual habits.

Firstly, the term "building" is employed in obv. 17 when the length is explained to be equal to the sum of the widths and an extra amount of *10*. It is not excluded that a constructive procedure is still intended, but in that case a mental construction is more plausible than an actual drawing. In any case, the formulation deviates from a normal usage which appears to be strongly bound up with specific procedures.

Secondly, a "counterpart" turns up in rev. 10 in a most unusual function. Normally, it is seen in length-width-problems (cf. YBC 6967, section V.1), when two sides containing a completed square are "laid down", for subtraction and ensuing addition of the *takīltum*.[176] In the present case, addition and subtraction

176 "Normally", but not exclusively, it is true. In TMS XIX, a number 1 is posed (in a "single false position") for the (real) "lenght" of the problem, and next also for its

of a semi-difference is still meant, but if a geometrical configuration is at all thought of, it is different, the "original" and the "counterpart" being opposing widths of a rectangle, which the addition and subtraction are to transform into a trapezium.

These peculiarities do not prevent a naive-geometric interpretation. Moreover, the "doubling" in obv. 23 suggests the use of a procedure related to a trick used in the two tablets VAT 7532 and VAT 7535 (both in MKT). The suggested procedure is shown in Fig. 16: The step of obv. 21f. corresponds to a scaling in horizontal direction (the first transformation, A→B). The repetition in obv. 23 is a genuine duplication, transforming the trapezium into a real rectangle (B→C), viz. a "surface (of a square) with 15 confrontations". The sequence of operations

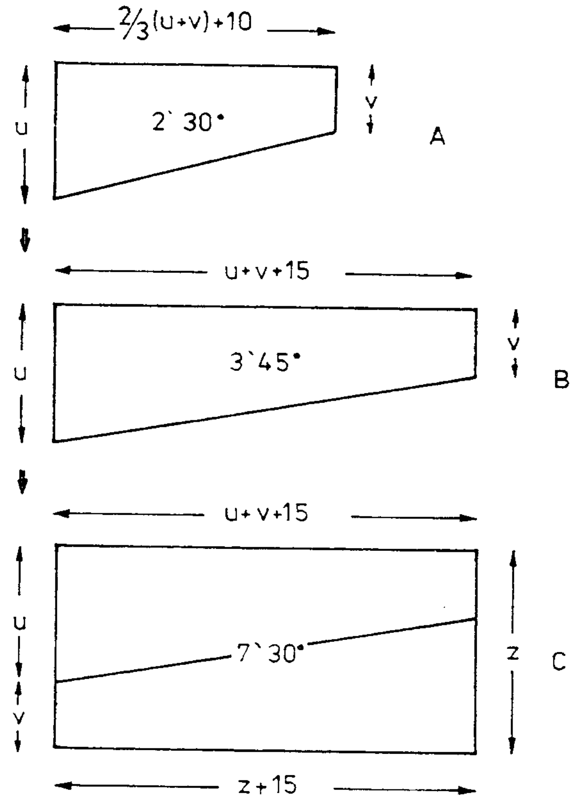


Figure 16. The geometrical interpretation of IM 52301 N° 2 suggested by the parallels in VAT 7532 and VAT 7535.

"counterpart" (TMS, 101, as corrected in von Soden 1964: 49), which in the following turns up to be its "basic representative". In TMS IX 40 (above, section VIII.3) as well as TMS XII 10 (TMS 79, as corrected in von Soden 1964: 49), and rev. 5 of the present text, the "original" and the "counterpart" form the usual geometric configuration, but already at the point where they are "made span" a supplementary square, not when the side of the completed square is found.

An occurrence in IM 55357, 10 (Baqir 1950) is still more deviant but need not occupy us here, as it has to do with a triangle.

is, however, remarkable. If the geometrical procedure had been performed physically, it would have been natural to make the very palpable doubling first, and the scaling afterwards. The actual sequence appears to indicate that a more purely arithmetical understanding of the underlying structure, where the sum of the widths is aimed at as an unknown (in the first transformation) before it is actually produced (in the second transformation).

The deviant use of the term "building" was already mentioned as an indication pointing in the same direction. The implications of the peculiar use of "counterpart" in rev. 10 are more indefinite, and the most that can be said is that an otherwise strict conceptual structure appears to be loosening, especially if we notice that the term is also used in a somewhat more orthodox way in rev. 5. The way the text regards the "equilateral" is, however, yet another indication that an arithmetical conceptualization is present: It is definitely no entity producing a square–it is something which "comes up", i.e. a numerical result.[177]

The awareness of a homomorphism between geometrical and arithmetical procedures need not have been greater with the author of the present text than with the authors of more orthodox, somewhat older texts. The latter, however, formulate themselves strictly within the geometrical conceptualization. This strictness of language has either been regarded as superfluous or has not been understood by the present author. In both cases it is justified to speak of a loosening of the conceptualizations and of an opening toward explicit arithmetical understandings.

X.2. Seleucid arithmetization: BM 34568 N° 9 (MKT III, 15)

Further developments of this opening toward arithmetic are seen in the algebra problems of the Seleucid era. A simple instance is found in BM 34568 N° 9, the very problem which was used in Chapter I to demonstrate the ambiguities of current translations. In transliteration and conformal translation, the text runs like this:

Obverse II

$x+y=14$ $x \cdot y=48$	1. Length *and* width accumulated[a]: 14, *and* 48 the surface. uš *ù* sag gar-[*m*]*a* 14 *ù* 48 a-šà
$(x+y)^2=3'\ 16^c$ $4 \cdot x \cdot y=3'\ 12^c$	2. The NAME[b] I *know* not. 14 steps of 14, 3' 16[c]. 48 STEPS[c] of 4, 3' 12[c]. MU nu-zu[ú] 14 a-rá 14 3, 16 48 GAM 4 3, 12
$(x-y)^2=(x+y)^2-4xy$ $=3'\ 16^c-3'\ 12^c$ $=4$	3. From[d] 3' 12[c] (to) 3' 16[c] go up[e]: 4 *remains*[f]. *What* STEPS *of what*[g] 3, 12-ta 3,[1]6 nim-*ma ri-ḫi* 4 *mi-nu-ú* GAM *mi-ni-i*

[177] The same expression is found in the contemporary and equally northern tablet Haddad 104 (al-Rawi – Roaf 1985) and in the late Old Babylonian or perhaps even early Kassite AO 17264 (MKT I 126). Db_2-146 (Baqir 1962), which is also contemporary with the present text, regards the "equilateral" as something which is to be "taken", presumably also as a numerical result.

$x-y=\sqrt{4}=2$	4. *shall I* GO[h] *so that*[i] 4? 2 STEPS of 2, 4. From 2 (to)
$(x+y)-(x-y)=14-2$	14 go up: 12 *remains*.
$=12=2y$	*lu*-rá-*ma lu* 4 2 GAM 2 4 2-ta 14 nim-*ma ri-ḫi* 12
$y=\frac{1}{2}\cdot 12=6$	5. 12 TIMES 30′, 6 the width. To[j] 2 *add*[k] 6: 8, 8 the
$x=(x-y)+y=2+6=8$	length.
	12 GAM 30 6 6 sag 2-še 6 *ta-ṭip-pi-ma* 8 8 uš

[a] "accumulated" translates GAR, which is certainly an abbreviation for gar-gar, not as in Old Babylonian texts a logogram for *šakānum*, "to pose".

[b] NAME translates MU, used logographically for *šūmum*. F. Thureau-Dangin's interpretation as a logogram for *aššum*, "since" (TMB, 59) is possible, but it does not fit the context. O. Neugebauer's interpretation "name" is, on the other hand, confirmed by the Susa text TMS IX.

[c] STEPS translates GAM, which in the contemporary mathematical table text MM 86.11.410 is used as a separation sign (see MCT, 15). In the present tablet, the sign appears to be used as a complete equivalent for a-rá, "steps of" (so also in the contemporary AO 6484 – MKT I, 96–99).

[d] "from" translates the Sumerian ablative-/instrumental suffix -ta.

[e] "go up" translates the Sumerogram nim, which in certain Old Babylonian texts was used as a substitute for íl∼*našûm*, "to raise", i.e. "to calculate by multiplication". Here the term appears in the original Sumerian meaning, used to describe a subtraction conceptualized as a counting process.

[f] "*remain*" translates *riāḫum*, "übrig bleiben".

[g] The first "how much" (*mīnum*) is a nominative, while the second is a genitive (*mi-ni-i*). So, the two factors in a product by GAM (and, as revealed by obv. I, 16f. of the same tablet, by a-rá) play different roles. It is this construction which has suggested my standard translation for a-rá (cf. section IV.3).

[h] "GO" translates rá, "to go" (TÙM in MKT). This supports the conclusions of notes c and g.

[i] "so that" translates the optative and precative partivle *lū* (also used to denote the precative form of the ideogram rá in the same line, "*shall I* GO").

[j] "to" translates the Sumerian terminative suffix -šè.

[k] "*add*" translates *ṭepûm*, "hinbreiten, auftragen; addieren", which in Late Babylonian had taken the place of *waṣābum*, "to append" (cf. von Soden 1964: 48a). In contradistinction to *waṣābum*, however, *ṭepûm* can be used as a symmetric term, *ṭepûm a together with b*. So, the modernizing connotations of the translation "to add" seem quite to the point.

First of all we observe that certain parts of the vocabulary are continuous with that of our Old Babylonian texts: "length", "width", "surface", "name", "steps of". All except "steps of" belong on the level of algebraic problems, not

on that of mere computation. We can therefore be sure that we are really confronted with a descendant of the Old Babylonian algebraic tradition, in spite of the silence of all sources between c. 1600 B.C. and c. 300 B.C.

The next observation will be that of thorough change on all levels, in spite of the continuity. It goes down to the choice of Sumerograms: nim, which in Old Babylonian texts designates a multiplication of the "raising" class, standing presumably for forms of *ullûm* (cf. note 39), is used now for the stepwise counting of a difference, presumably as a logogram for *elûm*. In part, at least, the Sumerianization of mathematical language appears not to have been continuous over the silent millenium.[178]

The discontinuous Sumerianization carries implications for the nature of the transmission, which appears to have taken place in a practitioners' environment rather than a scholarly institution. As far as the conceptual structure of Seleucid algebra concerns it has less to tell. Under the latter aspect, indeed, the absence of all traces of constructive thought and not least the purely arithmetical formulations are the most conspicuous features. Subtraction has become a straight counting process, instead of a concrete process described metaphorically in physical terms ("tearing out", "cutting off", etc.). Only one multiplicative operation is left, described by the term of multiplication tables, i.e., as a repeated counting, when not by the ideogram GAM, the separation sign used apparently as a purely visual symbol. Bisection is no special operation, but only a multiplication by 30′, and the square-root is explicitly asked for as the solution to the problem $x\cdot x=n$. Two additive processes appear to be present, but the one corresponding to "appending" can no longer be identity-conserving, since it is often, though not here, symmetrical with respect to the addends. No doubt, therefore, that the conceptualization of the problem is completely arithmetical.

As discussed at some length in chapter I, an arithmetical conceptualization does not exclude a geometrical method and justification. This combination is precisely what is found in al-Khwārizmī's justifications. A figure which would serve to solve the problem was shown in Fig. 2, and the same figure and a generalized version will in fact explain all problems of the tablet, except one dealing with alloying of metals and one concerned with a rectangle of known proportions (see Fig. 17). Moreover, even the more specious procedures are easily argued from the two all-purpose figures, and in one case, that of N° 13, O. Neugebauer feels obliged to have recourse to Fig. 17 B[179] in order to explain why the procedure is at all meaningful. On the other hand, several of the solutions are very difficult to follow unless one uses either geometric support or written, symbolic algebra – purely rhetorical methods will not do. It is therefore reasonable to assume that the method of Seleucid second-degree mathematics remained geo-

[178] Another case of re-Sumerianization is that of tab. In Old Babylonian mathematics, it was used as a logogram for *eṣēpum*, "to repeat"; in the present tablet (e.g. obv. I 2) it is used for *ṭepûm*, "to add". Both uses are in agreement with the general meaning of the Sumerian term; in their technical use, however, the two functions of the ideogram cannot be connected in any way, which excludes any continuous existence of tab as a mathematical term.

[179] Of course in symbolic transcription (MKT III 21). The important thing is that the entity $(l+w+d)^2$ cannot be avoided in the interpretation of the procedure.

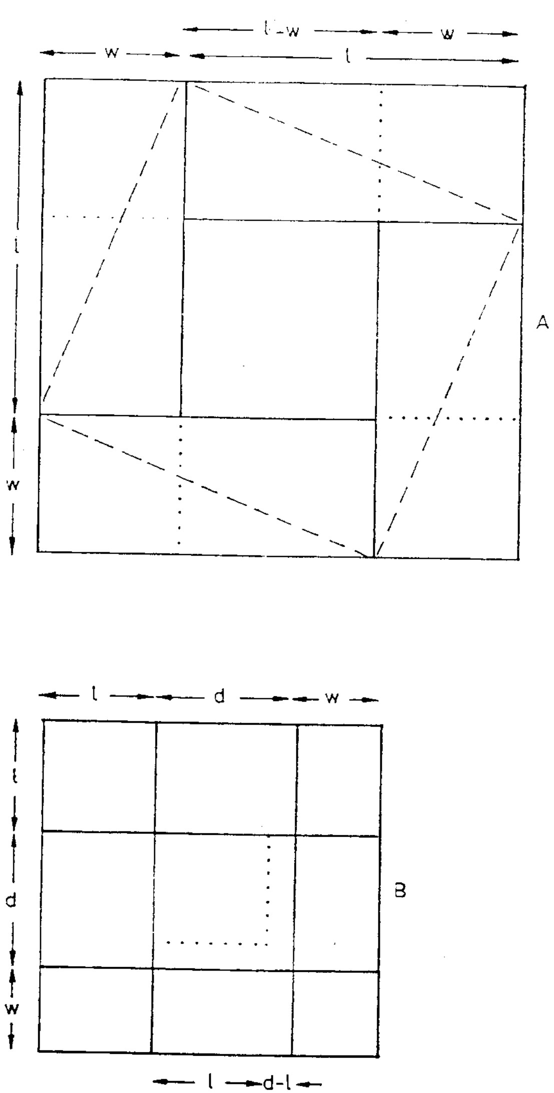


Figure 17. Two all-purpose figures which may support all the second-degree problem solutions of BM 34568. The upper figure will be recognized as a familiar justification of the Pythagorean theorem. For use of the lower figure, where d is the diagonal of a rectangle with length l and width w, one shall remember that the central square equals the sum of the upper left and the lower right square ($d^2 = l^2 = w^2$). In problem 12, the equality of the lower right square and the central gnomon will have to be used explicitly.

The upper figure is seen to contain Figure 14A, the one constructed for AO 8862 N° 3. It will be remembered (see above, note 138) that the same configuration appears to be used in two other Old Babylonian problems.

metric, in spite of the arithmetization of its conceptualization, though probably "synthetic" rather than analytically constructive.

It is tempting to see the arithmetical conceptualization as the final outcome of a natural process already begun during the late Old Babylonian period: Secular use of the same procedures would grind off everything superfluous and leave back only the essential structure, which is indeed arithmetical. Before accepting this as sole and sufficient explanation we should, however, be aware that another factor was also at work, and perhaps even a third circumstance should be taken into account.

The indubitable extra factor is the specific scholarly environment of Seleucid mathematics: The great astronomical centre of Uruk.[180] The enormous numerical calculations performed in this centre may well have made the local scribes more inclined toward arithmetical thought than less specialized practitioners of the algebraic art whoever they may have been. But as we shall see below, such practitioners must have existed.

The possible extra factor is cultural cross-fertilization. Seleucid Uruk was part of the Hellenistic melting-pot, and links back to Old Babylonian traditions should therefore not be taken to exclude combination with other links. In another branch of Seleucid mathematics, viz. mensurational geometry, a definite break with Old Babylonian methods and a striking parallel to Alexandrinian geometry is clearly visible.[181]

In the procedure of our problem there may also be a suggestion of cultural import. All corresponding Old Babylonian problems find the semi-sum and the semi-difference between length and width, even those which appear to make use of the same geometrical configuration. In the present case, the total sum and difference are found. There is no inherent reason for that change. In a group of more orthodox second-degree problems in the Seleucid tablet AO 6484, dealing with *igûm-igibûm*-pairs with known sum[182]—as far as mathematical structure concerns no different from the present problem—, we find indeed the traditional semi-sums and semi-differences, together with a terminology which is about as arithmetical as that of the present problem.[183]

[180] AO 6484, the other Seleucid tablet containing second-degree problems, was indeed written by Anu-aba-utēr, an early 2nd-century scribe from Uruk, known as possessor and writer of astronomical and other tablets. See the colophone in MKT I 99, and Hunger 1968: 40 (N° 92) and passim. If the algebraic tradition was really transmitted since the Old Babylonian period in an environment of "higher artisans", as suggested above, the circle of the Uruk astronomer-priests may be the setting where its re-Sumerianization took place.

[181] In Old Babylonian mensuration, the area of an irregular quadrangle had been found by the "surveyors' formula", as the product of "average length" with "average width" (see e.g. YBC 4675, in MCT 44f.). In the Seleucid tablet VAT 7848 (MCT 141) the height of a trapezium is calculated by means of the Pythagorean theorem, and everything goes exactly as in Hero's *Geometrica* 16, 17. New evidence suggests, it should be observed, that the development toward greater precision in mensuration may have taken place before the possible interaction with Greek geometry; indeed, unpublished Late Babylonian tablets contain the explicit calculation and use of the height of a triangle [Friberg (forthcoming) §§ 5.4e and 6.5].

[182] Rev. 10—27 (4 problems in total). In MKT I 98f.

[183] The subtraction of "surfaces" carries a *libbi*, "inside"; but the subtractive term itself

A purely autochthonous development would probably have affected the method of all isomorphous problems similarly. It is therefore plausible that the specific methods of BM 34568 were introduced together with a specific cluster of length-width-diagonal-problems during the dialogue of scientific cultures.

It is not possible to identify the eventual interlocutor. Similar interest are found in China, in the *Nine Chapters on Arithmetic.*[184] But they are also found in the Graeco-Roman world[185], and in neither case are the similarities complete nor fully convincing. Furthermore, the Hellenistic era was one of wide-range cultural connections, from China to Magna Graecia. The suggestive similarities can at most be taken as indications that mutual inspiration took place, and that Babylonia was probably not the only focal point for "algebraic" investigations of geometric figures.

X.3. Babylonian influence in Greek mathematics?

The hypothetical foreign inspiration of Seleucid algebra is difficult to trace precisely. So are also the possible inspirations flowing the other way during Antiquity and the early Middle Ages. Certain suggestions can be found, however, in Greek sources pointing to inspiration though hardly to direct descendency.

The idea of inspiration from Babylonian algebra to Greek "geometric algebra", i.e. the geometry of "Elements II" etc., is as old as the discovery of Babylonian second-degree algebra. Since the late 1960es it has been submitted to severe criticism[186], mainly because the Greek geometry of areas is a coherent structure of its own which is not adequately explained as a "translation" of an arithmetico-rhetorical algebra, of which it is neither an isomorphic nor a homomorphic mapping.

A naive-geometric reinterpretation of Babylonian algebra changes much of the foundation of the debate.[187] If we recognize further that the structure of Greek geometry is the result of a process and not identical with the structure of its possible inspirations, the question of Babylonian inspiration of Greek mathematics is completely open again.

This is not the place for a thorough investigation of the problem, which I approach elsewhere.[188] I shall just point to the observation which put me on the track. The much-discussed term δύναμις has given rise to precisely the same ambiguities as the Babylonian *mithartum*. In some contexts it seems to mean "square-root" or "side of square", in others it is the square itself. As in the Bab-

is lal, "diminish", and the addition is expressed simply by *u*, "and", and tab, "add". Multiplication is comprehended as "going steps".

[184] Translated by Vogel (1968—the relevant problems are found pp. 90—103).

[185] One source is a Greek papyrus from the 2nd century A.D. (Rudhardt 1978, cf. Sesiano 1986). Another is a Latin *Liber podismi* (latest edition in Bubnov 1899: 510—516), dating perhaps to the 4th century A.D. and based apparently on Alexandrian sources. One of its problems (ibid. 511f.) deals with a right triangle, for which the hypotenuse and the area are known. The solution is of "Seleucid" type, making use of total sum and total difference.

[186] I shall only refer to Szabó 1969; Mahoney 1971; and Unguru—Rowe 1981.

[187] See my 1983 (review of Unguru—Rowe 1981).

[188] See my 1988.

ylonian case, the apparent ambiguities are eliminated if we read the term as "a square identified by (and hence with) its side". The normal Greek habit is to identify a figure with its area; as with us, a square designated τετράγωνος has a side and is its area. The δύναμις is thus a foreign flower in the Greek conceptual garden.

Investigation of a variety of mostly early sources suggests that the term was not only used in theoretical geometry but also by calculators, seemingly in connection with some sort of algebraic activity, an earlier stage of the tradition behind Diophantos. Links to the theory of figurate numbers are also suggested, and hence to a pebble-abacus-representation of naive-geometric procedures (cf. above, the end of chapter VI).[188a]

Another possible line of transmission of Babylonian influence goes to the pre-Diophantine algebraic tradition. I have already pointed at the similar ways in which the Babylonians and Diophantos deal with non-normalized problems, and other similarities could be found in that tiny part of Diophantos' "Arithmetica" which possesses cuneiform parallels. Such similarities are, however, fairly inconclusive, since the subject-matter itself restricts the range of possible procedures strongly. Supplementary evidence may, however, be hidden in a much-discussed term of the "Arithmetica", the πλασματικός, which occurs in I.xxvii, I.xxviii and I.xxx of the surviving Greek part, and in the Arabic IV.17, V.19 and V.7. In the Greek text, it seems to be the diorism, i.e. the condition for solvability which is called πλασματικόν, while the Arabic passages speak of the whole problem as belonging to the class of *al-muhayya'ah.*[189]

The Greek term derives from πλάσσω, "to form", "to mold", etc., and it is related to πλάσμα, "anything formed or molded, image, figure" etc. (GEL 1412a). Because of this etymology and the Greek passages alone, P. Ver Eecke suggested it to mean that the diorism can be demonstrated geometrically.[190] Since a reference to Euclidean geometry fits badly to the distribution of the term in the Arabic books, both editors of the Arabic text have looked for alternative ways to get a meaning of the term in its actual contexts.[191] Here again, however, the naive-geometric view-point changes the basis of the question. We already know a πλάσμα, a fixed figure or "mold" on which the diorisms of the three Greek passages can be seen immediately; viz. the upper square in Fig. 17 (quartered as in Fig. 14, since Diophantos uses semi-sums and semi-differences). Moreover, the

[188a] In this connection it may be of some interest, but is of course inconclusive, that the method of BM 34568 N° 9 is better suited for treatment by pebbles than the traditional semi-sum/semi-difference procedure, which fails if the sum or, equivalently for integers, the difference is odd.

[189] The first Arabic passage is grammatically impossible as it stands. Rashed (1984: III 27) prefers a minimal correction, which makes the term an epithet to a number. Sesiano (1982: 99 note 48) makes a more radical emendation, in order to obtain agreement with a backward reference in the next passage and with his own interpretation of the term. The first but not the second of these considerations seems compelling to me, which makes me accept that part of Sesiano's correction which makes IV 17 a parallel to IV 19 (whence also to V.7).

[190] Ver Eecke 1926: 36 note 6. There is no reason to go further into the details of his explanation.

[191] Rashed 1984: 133—138; Sesiano 1982: 192f.

diorism of the Arabic V.7 can be seen on the three-dimensional analogue of the same figure.

The diorisms of the Arabic IV 17 and IV 19 are of a different character, involving factorizations of the sides of cubes. There are no direct links to specific Babylonian material. On the other hand, certain techniques used for the computation of large reciprocal tables and the techniques of scaling are akin to the Diophantine procedure. Since at least the Arabic text does not claim that these and none but these problems possess a distinctive mathematical quality but only states that they belong to a certain pre-established bunch of problems possessing the quality, we should perhaps interprete the term as designating problems the feasibility of which is seen by certain naive-geometric procedures, not necessarily by Diophantos but at least by the people who established the bunch. The interpretation is not compelling, nor is however any rival explanation. A hint of a Babylonian connection may–but need not–hide behind the term and the concept.

X.4. A direct descendant: *Liber mensurationum*

If inspirations from Babylonian algebra to Greek mathematics can only be traced indirectly, through the combination of many sorts of roundabout evidence, influences in Medieval Islamic mathematics are direct and easily verified.

Once more, I shall only sketch the basis of the argument, since I deal with the matter in detail elsewhere.[192] The central source is a Latin translation made by Gherardo di Cremona in the 12th century from an Arabic original due to one otherwise unidentified Abū Bakr, the *Liber mensurationum*.[193] The first parts of the work deal with squares and rectangles (the later parts, related to Alexandrian practical geometry, do not concern us here). It was already noticed by H. L. L. Busard in his edition that the work shares many problem-types and even the coefficients of certain problems with Babylonian algebra (making no distinction between Old Babylonian and Seleucid material). This, however, is not conclusive. Starting from the simplest cases you will necessarily hit upon many of the same problem-types when progressing toward more complex algebraic problems, and if you prefer, e.g., the second-simplest to the simplest Pythagorean triangle, your numbers will be 6, 8 and 10.

The first decisive observation is that many problems are solved twice, first by a method given no specific name and hence to be regarded as the normal, fundamental method, and next by *aliabra*, obviously a term meant to render the Arabic *al-jabr*. In a general sense of the word, both methods are equally algebraic. *Aliabra*, however, refers directly to the fundamental cases known from al-Khwārizmī. It is hence the rhetorical discipline known from al-Khwārizmī and ibn Turk[194] and also referred to by Thābit ibn Qurra in his "Rectification of the cases of *al-jabr*".[195] In several cases, the numerical steps of the fundamental method and the alternative by *aliabra* are identical. The difference between the two must therefore be one of representation and conceptualization.

[192] See my 1986.
[193] Critical edition by Busard (1968).
[194] See Sayılı 1962.
[195] See Luckey 1941.

The next observation is that the discursive organization of the descriptions of the "fundamental" procedures coincides down to the choice of grammatical tense and person and to the use of certain standard phrases ("since he has said"; "may your memory retain") with the familiar structure of Old Babylonian texts. The procedures are also often those known from the Old Babylonian texts, e.g. the "change of variable" of AO 8862 N° 1. The standard length-width-problem is solved by means of semi-sum and semi-difference, showing that the connection of the text is really directly to the Old Babylonian tradition, bypassing the Seleucid astronomical school.

A closer look at the vocabulary shows that the conceptual distinctions known from the classical Old Babylonian tradition are not respected completely. So much remains, however, that we have good reasons to believe that a naive-geometric method is still behind the numerical algorithms described in the text. A final "See" after many procedure-descriptions indicates that the original has indeed contained (naive-)geometric justifications of the methods.[196]

These observations are the main but not the sole reasons to see the fundamental approach of the text as a direct continuation of an Old Babylonian naive-geometric tradition, which must then have been alive until the Arabic original was written, probably not much later than A.D. 800. Even in Abū Kāmil's *Algebra*, dating from c. A.D. 900, an alternative to the normal *al-jabr* procedure is sometimes offered[197] which contains the typical Old Babylonian steps, though in arithmetico-rhetorical disguise. More striking is, however, a passage in Abū'l Wafā''s *Book on What is Necessary from Geometric Construction for the Artisan*, written shortly after A.D. 990. In chapter 10, prop. 13, the author tells that he has taken part in certain discussions between "artisans" and "geometers", apparently regarded as coherent groups. Confronted with the problem of adding three equal geometric squares, the sum also being a square, the artisans proposed a number of solutions, "to some of which were given proofs", proofs which turn out to be of cut-and-paste character. The geometers too had provided a solution in Greek style, but that was not acceptable to the artisans, who claimed a concrete re-arrangement of parts into which the original squares could be cut.[198]

[196] One may wonder that so many linguistic observations can be made on a Medieval Latin translation. The reason is that Gherardo's translation appears to be extremely literal, reflecting even some peculiarities in the original usage which could easily have been straightened without loss of mathematical substance.
[197] See Levey 1966: 94, 96.
[198] See Krasnova 1966: 115ff. This Russian translation is the only printed version of the work, although selections and paraphrases from incomplete manuscripts have been published by Woepcke (1855) and Suter (1922: 94–109). Though not algebraic the whole treatise is highly interesting as an eclectic merger between a Near-Eastern naive-geometric tradition and Greek apodictic geometry. Abū'l-Wafā's treatise is a main source for the establishment of a connection between the cut-and-paste technique and the later theory of partition of figures. Another work of possible interest in this connection is a short treatise on the Pythagorean theorem written by Thābit ibn Qurra (description in Sayılı 1960, Arabic text and Turkish translation in Sayılı 1958). The first part of the treatise describes two proofs of the theorem by means of *al-tafṣīl wa'l-waṣl*, "partition and combination" (Sayılı 1958: 535 l. 7). The figures are, however, different from those connected to the Babylonian tradition; they look rather as generalizations of that used by Socrates in Plato's *Meno*, and the method is indeed described

A striking feature of the *Liber mensurationum* is the recurrence of problems adding or subtracing the four sides of a rectangle or square from the area (or reversely); other multiples of the sides do not occur. In a problem collection derived from surveying and surveyors' interest this comes as no great surprise. As in Old Babylonian mathematics, inhomogeneous second-degree-problems could only arise as artificial constructions, and most easily as recreational problems. But a funny problem in surveying is one which adds the area and all four sides of a square field rather than one which (like BM 13901 N° 2) adds $^{2}/_{3}$ of the area and $^{1}/_{3}$ of the side. Recreational problems in general are not characterized by mere complexity and artificiality but first of all by striking coincidences. This observation is part of the evidence for the above claim that the aberrant problem 23 from BM 13901 (section V.4) was taken over from a surveyors' tradition and adopted into the school tradition, perhaps even as the source for the interest in inhomogeneous second-degree "algebra".[199]

As regards the methods of the *Liber mensurationum*, it is noteworthy that the trick used in AO 8862, problems 1 and 2, is used time and again. These early problems, we remember, were formulated as "surveying anecdotes". Their methodological affinity with the late surveying tradition can thus be regarded as supplementary evidence that Old Babylonian school "algebra" and the *Liber mensurationum* both derive from a common, older mensuration tradition.

In chapter I I used al-Khwārizmī's naive-geometric justifications of his algorithms as a pedagogical device, in order to demonstrate what naive geometry would look like. At the present stage of the investigation it turns out that the old naive-geometric tradition was still alive when al-Khwārizmī wrote his seminal compendium on algebra. We can hardly assume that he invented anew a technique which was widely practiced around him, and we can therefore be confident that his justifications were direct descendants of those of the Old Babylonian calculators. We may guess that even his arithmetico-rhetorical *al-jabr* derives ultimately though highly transformed from the same source, but there we have no direct evidence. Through his justifications, however, we know that the ancient techniques were passed on to Medieval Islam and to the early European Renaissance, and hence to the modern world.

as „Socratic" by Thābit; not being able to follow the text, I am thus not sure about its implications.

199 If this hypothesis is correct, the tradition will have been carried by Akkadian speakers, according to the explicitly Akkadian *eqlam* introducing BM 13901 N° 23. This fits "the Akkadian" method as a name for the quadratic completion (TMS IX, see section XIII.3). It also agrees with the Akkadian language of the whole Old Babylonian mathematical tradition which, as observed repeatedly above, is visible even in its use of quasi-Sumerian logograms. Old Babylonian school mathematics was—like omen literature which is likewise written in Akkadian—new as a school tradition, but it may well have older oral roots. A Sargonic tablet bisecting a trapezium [Friberg (forthcoming), section 5.4.K] suggests that it goes back at least to the 23d century B.C. The present hypothesis on the relation between Old Babylonian school mathematics and the surveyors' tradition is argued in somewhat more detail in Høyrup 1989a: 28f.

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